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A random matrix theory of decoherence

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Abstract. Random matrix theory is used to represent generic loss of coherence of a *fixed* central system coupled to a *quantum-chaotic* environment, represented by a random matrix ensemble, via random interactions. We study the average density matrix arising from the ensemble induced, in contrast to previous studies where the average values of purity, concurrence and entropy were considered; we further discuss when one or the other approach is relevant. The two approaches agree in the limit of large environments. Analytic results for the average density matrix and its purity are presented in linear response approximation. The two-qubit system is analysed, mainly numerically, in more detail.

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Contents

1. Introduction

Decoherence has received increasing attention, since it is not only related to the problems of measurement and emergence of classical behaviour [1]–[4] but has become one of the main stumbling blocks for quantum information solutions. Under usual circumstances decay of coherence is exponential and this has been confirmed by experiment [5, 6]. Exponential decay of coherence typically holds if the Heisenberg time of the environment is too long to provide a relevant timescale, i.e. if decoherence is complete long before this time. As the global environment is always present and has, for all practical purposes, infinite Heisenberg time, our discussion seems spurious. Yet this is not always the case. Quantum information tasks require sufficiently low temperatures, high vacuum, good screening, carefully selected states, etc that the global environment can be neglected to first approximation. We are left with a near environment consisting of only a few degrees of freedom which are involved in the control of the operations of the quantum gates. This near environment may have very specific properties, but is often governed by dynamics that might pass for 'quantum chaotic', or whose dynamics we ignore. The purpose of this paper is to provide a generic description of decoherence which covers such situations, as well as the standard ones.

This requires modelling of the environment, while avoiding a complete microscopic description. More specifically, we describe a generic environment with 'quantum chaotic' or unknown dynamics by random matrix theory (RMT). We may emphasize that the original meaning of quantum chaos can be strictly viewed as the study of quantum properties characteristic of systems whose classical analogue is chaotic. The main signature is reflected by statistical properties which are mathematically well described by RMT [7]–[11]. Many authors use the concept 'quantum chaotic' almost synonymously to 'having RMT properties' even when a classical analogue is not obvious. It may be worthwhile to remember that RMT has also often

been justified as reflecting our ignorance of a system and can be derived related to minimum information arguments [12]–[14], which may give intuitive understanding of the success of applications of RMT to uncontrolled environments. Some basics of quantum chaos and RMT will be given in appendix A.

A first step toward the introduction of RMT into problems of decoherence is the pioneering work of Lutz and Weidenmueller [15, 16] in which they propose a RMT description that reproduces results of the Caldeira–Leggett model. Using banded matrices they show that RMT works in this standard situation. We shall see that this is closely related to choosing a very dense spectrum in the environment, which sends the Heisenberg time to infinity. More generally, whenever decoherence is strong before the Heisenberg time of the environment is reached, we can use a Markovian approximation and recover a master equation. To show that our model yields the usual results in this regime, we shall explicitly derive the corresponding master equation and look into some resulting properties of the system.

While this argumentation is very reasonable for the far environment, the near environment can have level densities comparable to those of a central system, and this is where RMT models can produce new results. A model to study such effects was first proposed in [17, 18] and further developed in [19, 20]. These models are closely related to one developed for fidelity decay [21], which was successfully tested in experiments [22]–[24]. For decoherence, the models were compared to numerics with a spin chain environment [25].

In these models purity was calculated analytically in linear response approximation as averaged over the random matrices. For the purpose of quantum information, this was quite sufficient, as only high purities are needed. Yet, the beauty is that these RMT models are sometimes exactly solvable by super-symmetric techniques. Attractive examples are the solutions for the decay of the fidelity amplitude [26]–[29]. These techniques nowadays are still limited to the calculation of two-point functions. Purity is obviously of higher order, and thus we should look for an amplitude. Coherences, i.e. off-diagonal elements of the density matrix, are good candidates, as their decay is often used to describe decoherence [6]. This suggests that we must consider the average density matrix, something that has not been done so far.

Actually, in the formulation and use of RMT models for decoherence a basic question has not been addressed. If we consider the evolution of some initial state by an ensemble of Hamiltonians we get an ensemble of density matrices for the central system. Normally, we would consider the average density matrix as the basic quantity from which observables, such as purity or concurrence, should be calculated. However, since these quantities are nonlinear in the density matrix, their average (over the ensemble of density matrices) is different from their value computed from the average density matrix. It is the particular application which determines which kind of averaging must be chosen. We shall discuss this in section 5, but we will find in the previous sections that for large environments the quantities we analyse, namely purity, entropy and concurrence, converge to the same result whether explicitly averaged or taken for the average density matrix.

We first formulate the random matrix model we intend to use and derive a few basic relations. Some obvious conditions the model must fulfil are presented before proceeding to calculate the average density matrix in linear response approximation. For the particular case of one qubit as a central system we outline how to obtain the exact answer with super-symmetric theory. Similar results for the average density matrix are presented in what we earlier called spectator configuration [30]. This configuration implies a central system consisting of two parts, one of which has no interaction with the other or with the environment, but is entangled with

the other part of the central system. This configuration has proven very useful to describe a composite central system with high purity [31]. We next concentrate on two-qubit central systems in such a spectator configuration. For these we display some properties of the density matrix ensemble including statistics of the eigenvalues as well as further results on a curious relation to Werner states shown in previous papers [20, 25]. In section 5, we discuss in more detail the meaning of an ensemble of density matrices. This discussion will include both situations where one should use the average density matrix to calculate nonlinear functionals and others where this is not appropriate. This leads us to conclusions and an outlook on the relevance of more realistic RMT environments.

2. Random matrix models

We shall start with a central system that does not display any particular structure, and thus the entire system will be bi-partite. Decoherence will be viewed as the entanglement of the central system with the environment after unitary evolution of the entire system. As we wish to describe a near environment with quantum chaotic or RMT features, we neglect the far environment entirely, as indeed we shall throughout this paper, i.e. we assume, that decoherence resulting from the far environment occurs on a much longer timescale.

The Hilbert space is divided into two parts corresponding to the central system $\mathcal{H}_c$ and to the environment $\mathcal{H}_e$. The dimensions of those subspaces are denoted by $m = \dim(\mathcal{H}_c)$ and $N = \dim(\mathcal{H}_e)$, respectively. Eventually, the case $N \to \infty$ will be particularly important, but this limit will be taken for fixed Heisenberg time. For a factorized initial state of the form

$$\varrho_0 = \varrho_c \otimes \varrho_e, \tag{1}$$

we solve the von Neumann equation

$$\mathrm{i}\,\partial_t\, \varrho(t) = H_\lambda\, \varrho(t) - \varrho(t)\, H_\lambda, \tag{2}$$

where the Hamiltonian is of the form

$$H_\lambda = H_c + H_e + \lambda\, V, \tag{3}$$

such that $H_0 = H_c + H_e$. Note that at this point we do not require the initial state to be pure. It may well be a product of two mixed states ϱ_c and ϱ_e. In the following step, we express the forward evolution of the whole system in terms of echo dynamics. Echo dynamics implies that we consider a forward time evolution which includes the coupling between the central system and the environment and a backward evolution, which is driven by the internal dynamics of the environment and central system alone. The latter will not modify the entanglement between the two subsystems. This is useful, because we now can apply the linear response approximation to the echo-dynamics, which results in a much larger range of validity of this approximation. A systematic application of the interaction picture yields a similar simplification. Hence, we write:

$$\varrho(t) = \mathrm{e}^{-\mathrm{i}H_\lambda t}\, \varrho_0\, \mathrm{e}^{\mathrm{i}H_\lambda t}, \tag{4}$$

where we have set $\hbar = 1$. In order to obtain the echo dynamics, we evolve the system backwards in time with the Hamiltonian H_0:

$$\varrho_M(t) = M_\lambda(t)\, \varrho_0\, M_\lambda(t)^\dagger, \quad M_\lambda(t) = \mathrm{e}^{\mathrm{i}H_0 t}\, \mathrm{e}^{-\mathrm{i}H_\lambda t}. \tag{5}$$

This may be interpreted as an evolution with the echo-operator $M_\lambda(t)$ [19]. The evolution of the central system is then given by

$$\varrho_{\rm c}(t) = {\rm tr}_{\rm e}\left[\,\varrho(t)\,\right] = {\rm tr}_{\rm e}\left[\,(u_{\rm c}\otimes u_{\rm e})\,\varrho_M(t)\,(u_{\rm c}^\dagger\otimes u_{\rm e}^\dagger)\,\right], \tag{6}$$

where $u_{\rm c} = \exp(-{\rm i}H_{\rm c}t)$ and $u_{\rm e} = \exp(-{\rm i}H_{\rm e}t)$. This expression for $\varrho_{\rm c}(t)$ may be further simplified:

$$\varrho_{\rm c}(t) = u_{\rm c}\ {\rm tr}_{\rm e}\left[\,(\mathbb{1}_{\rm c}\otimes u_{\rm e})\,\varrho_M(t)\,(\mathbb{1}_{\rm c}\otimes u_{\rm e}^\dagger)\,\right] u_{\rm c}^\dagger = u_{\rm c}\,\tilde{\varrho}_{\rm c}(t)\,u_{\rm c}^\dagger, \tag{7}$$

where $\tilde{\varrho}_{\rm c}(t) = {\rm tr}_{\rm e}[\varrho_M(t)]$ is the density matrix of the central system in the interaction picture. Note that for operators of the form $\mathbb{1}_{\rm c}\otimes u_{\rm e}$ the cyclic permutations are allowed even though we are dealing with a partial trace.

Linear response approximation. In the linear response approximation, the echo-operator $M_\lambda(t)$ reads as

$$M_\lambda(t) \approx \mathbb{1} - {\rm i}\lambda\, I(t) - \lambda^2 J(t), \tag{8}$$

where

$$I(t) = \int_0^t {\rm d}\tau\ \tilde{V}(\tau), \quad J(t) = \int_0^t {\rm d}\tau \int_0^\tau {\rm d}\tau'\ \tilde{V}(\tau)\,\tilde{V}(\tau'), \tag{9}$$

and $\tilde{V}(t) = \exp({\rm i}H_0t)\ V\ \exp(-{\rm i}H_0t)$. Starting from equation (8), we obtain the density matrix of the central system averaged over the coupling matrix V (this average will be denoted by angular brackets). Subsequently, we may also average over the environmental Hamiltonian $H_{\rm e}$. However, for the linear response result as such, this is not necessary. In the interaction picture, we obtain:

$$\begin{aligned}\langle\tilde{\varrho}_{\rm c}(t)\rangle &\approx {\rm tr}_{\rm e}\,\langle\,(\,\mathbb{1} - {\rm i}\lambda\, I(t) - \lambda^2 J(t)\,)\,\varrho_0\,(\,\mathbb{1} + {\rm i}\lambda\, I(t) - \lambda^2 J(t)^\dagger\,)\,\rangle \\ &\approx \varrho_{\rm c} - \lambda^2(\,\langle A_J\rangle - \langle A_I\rangle\,),\end{aligned} \tag{10}$$

where we have used the notation

$$A_J = {\rm tr}_{\rm e}\left[\,J(t)\,\varrho_0 + \varrho_0\,J(t)^\dagger\,\right], \quad A_I = {\rm tr}_{\rm e}\left[\,I(t)\,\varrho_0\,I(t)\,\right]. \tag{11}$$

Note that $I(t)$ is self-adjoint, while $J(t)$ is not. According to equation (7), this yields for the time evolution of the central system:

$$\langle\varrho_{\rm c}(t)\rangle = u_{\rm c}(t)\,\langle\tilde{\varrho}_{\rm c}(t)\rangle\,u_{\rm c}(t)^\dagger. \tag{12}$$

The averaging procedure itself as well as the different possible statistical assumptions, will be discussed below.

Purity. We can now obtain the purity of the average density matrix of the central system. In view of (7), we find for the purity of the averaged state $\bar{\varrho}_{\rm c}(t)$:

$$P(t) = {\rm tr}\,\langle\varrho_{\rm c}(t)\rangle^2 = {\rm tr}\big[\,u_{\rm c}\,\langle\tilde{\varrho}_{\rm c}(t)\rangle\,u_{\rm c}^\dagger\,u_{\rm c}\,\langle\tilde{\varrho}_{\rm c}(t)\rangle\,u_{\rm c}^\dagger\,\big] = {\rm tr}\,\langle\tilde{\varrho}_{\rm c}(t)\rangle^2. \tag{13}$$

The purity may be more explicitly written in terms of the average state of the full system (in the interaction picture) as

$$P(t) = {\rm tr}_{\rm c}\,(\,{\rm tr}_{\rm e}\,\langle\varrho_M(t)\rangle\ \,{\rm tr}_{\rm e}\,\langle\varrho_M(t)\rangle). \tag{14}$$

Note that in this expression, we cannot exchange the order of the partial traces, since $\langle\varrho_M(t)\rangle$ is no longer a pure state in the product Hilbert space $\mathcal{H}$ of central system and environment. This

is the reason why we cannot simply adopt the calculations in [30] to the present case. In linear response approximation we find with the help of (10)

$$\begin{aligned} P(t) &\approx \mathrm{tr}(\,\varrho_\mathrm{c}^2\,) - \lambda^2\,\mathrm{tr}(\varrho_\mathrm{c}\,\langle A_J\rangle + \langle A_J\rangle\,\varrho_\mathrm{c} - \varrho_\mathrm{c}\,\langle A_I\rangle - \langle A_I\rangle\,\varrho_\mathrm{c}\,) \\ &\approx \mathrm{tr}(\varrho_\mathrm{c}^2\,) - 2\lambda^2\,\big[\,\mathrm{tr}(\langle A_J\rangle\,\varrho_\mathrm{c}) - \mathrm{tr}(\langle A_I\rangle\,\varrho_\mathrm{c})\,\big]. \end{aligned} \tag{15}$$

Since A_J, A_I and ϱ_c are all Hermitian operators, these expectation values are real.

Note the difference to the approach chosen in [17, 25, 30, 31], where we computed the average purity rather then the purity of the average density matrix. This difference will be discussed as we proceed and will be taken up in detail in section 5.

Statistical (random matrix) assumptions. We divide the statistical assumptions employed into two parts. A basic part which is invoked when we compute the density matrix of the central system, and an optional part, which makes stronger assumptions on the dynamics of the environment in order to obtain more explicit results.

- We assume that V is taken from the Gaussian unitary ensemble (GUE). This allows to work in the eigenbasis of H_0, where H_0 is diagonal. The transformation into this basis is necessarily a unitary transformation, such that the ensemble for V remains unchanged. In addition, due to the separability of $H_0 = H_\mathrm{e} + H_\mathrm{c}$, the eigenbasis may be chosen as the product basis $|ij\rangle$ of eigenstates of H_c and H_e, respectively.
- The choice of H_c is quite arbitrary, as long as it remains fixed. In contrast, we eventually assume H_e to be randomly chosen from an appropriate ensemble, say one of the classical ensembles. In that case, the two-point form factor $b_2(t)$ [32] is the only characteristic quantity of the spectrum. The two-point form factor may well be left unspecified. However, as specific examples we may consider Poisson spectra ($b_2(t) = 0$), as well as spectra resulting from GUE and Gaussian orthogonal ensemble (GOE) [32].

2.1. Average density matrix of the central state

Here, we calculate the behaviour of the central system, when averaging over the random-matrix coupling to the environment. The state of the central system is described by the density matrix $\langle\varrho_\mathrm{c}(t)\rangle$. In view of equations (10) and (12), we start by calculating the terms $\langle A_J\rangle$ and $\langle A_I\rangle$.

$$\langle A_J\rangle = \int_0^t \mathrm{d}\tau \int_0^\tau \mathrm{d}\tau'\ \mathrm{tr}_\mathrm{e}\big[\,\langle\tilde V(\tau)\,\tilde V(\tau')\rangle\,\varrho_0 + \varrho_0\,\langle\tilde V(\tau')\,\tilde V(\tau)\rangle\,\big], \tag{16}$$

where

$$\begin{aligned} \langle\tilde V(\tau)\,\tilde V(\tau')\rangle &= |ij\rangle\,\mathrm{e}^{\mathrm{i}\,(E_{ij}\tau - E_{mn}\tau')}\,\big\langle V_{ij,kl}\,V_{kl,mn}\big\rangle\,\mathrm{e}^{-\mathrm{i}\,E_{kl}(\tau-\tau')}\,\langle mn| \\ &= |ij\rangle\sum_{kl}\mathrm{e}^{-\mathrm{i}\,(E_{kl}-E_{ij})(\tau-\tau')}\,\langle ij| = \boldsymbol{C}_\mathrm{c}(\tau-\tau')\otimes\boldsymbol{C}_\mathrm{e}(\tau-\tau'). \end{aligned} \tag{17}$$

We use the implicit summation convention for indices appearing more than once. We use indices i, k, m for eigenstates and eigenenergies of H_c and j, l, n for eigenstates and eigenenergies of H_e. The notation E_{ij} stands for $E_{ij} = E_i + E_j$, where E_i is an eigenenergy of H_c and E_j is an eigenenergy of H_e. The diagonal matrices $\boldsymbol{C}_\mathrm{c}$ and $\boldsymbol{C}_\mathrm{e}$ are defined in analogy to [30] with the only difference that we use the subscript c instead of 1 to refer to the central system:

$$\boldsymbol{C}_\mathrm{c}(\tau) = \sum_{ik}|i\rangle\,\mathrm{e}^{-\mathrm{i}\,(E_k-E_i)\tau}\,\langle i|, \quad \boldsymbol{C}_\mathrm{e}(\tau) = \sum_{jl}|j\rangle\,\mathrm{e}^{-\mathrm{i}\,(E_l-E_j)\tau}\,\langle j|. \tag{18}$$

The separable result in equation (17) is due to the choice of the GUE as random matrix ensemble for the coupling. In the GOE case, we would obtain an additional term, which breaks the separability.

$$\langle A_J\rangle = \int_0^t \mathrm{d}\tau \int_0^\tau \mathrm{d}\tau' \left[C_\mathrm{e}(\tau-\tau')\, \boldsymbol{C}_\mathrm{c}(\tau-\tau')\, \varrho_\mathrm{c} + \text{herm. conjg.} \right], \tag{19}$$

where $C_x(\tau) = \mathrm{tr}_x[\boldsymbol{C}_x(\tau)\,\varrho_\mathrm{x}]$ for $x = \mathrm{c}, \mathrm{e}$, again in full analogy to [30]. For the second term $\langle A_I\rangle$, we obtain

$$\langle A_I\rangle = \int \int_0^t \mathrm{d}\tau \mathrm{d}\tau'\ \mathrm{tr}_\mathrm{e}\left[\langle \tilde{V}(\tau)\,\varrho_0\,\tilde{V}(\tau')\rangle \right], \tag{20}$$

where

$$\begin{aligned}
\mathrm{tr}_\mathrm{e}\left[\langle \tilde{V}(\tau)\,\varrho_0\,\tilde{V}(\tau')\rangle \right] &= |i\rangle\, \mathrm{e}^{\mathrm{i}(E_{ij}-E_{kl})\tau}\ \langle V_{ij,kl}\, V_{mn,pj}\rangle\ \varrho^\mathrm{c}_{km}\ \varrho^\mathrm{e}_{ln}\ \mathrm{e}^{\mathrm{i}(E_{mn}-E_{pj})\tau'}\ \langle p| \\
&= |i\rangle\, \mathrm{e}^{\mathrm{i}(E_{ij}-E_{kl})\tau}\ \varrho^\mathrm{c}_{km}\ \varrho^\mathrm{e}_{ln}\ \delta_{ip}\delta_{km}\delta_{ln}\ \mathrm{e}^{\mathrm{i}(E_{mn}-E_{pj})\tau'}\ \langle p| \\
&= |i\rangle\, \mathrm{e}^{\mathrm{i}(E_{ij}-E_{kl})\tau}\ \varrho^\mathrm{c}_{kk}\ \varrho^\mathrm{e}_{ll}\ \mathrm{e}^{\mathrm{i}(E_{kl}-E_{ij})\tau'}\ \langle i| \\
&= C_\mathrm{e}(\tau'-\tau)\ |i\rangle\, \mathrm{e}^{-\mathrm{i}(E_k-E_i)(\tau-\tau')}\ \varrho^\mathrm{c}_{kk}\ \langle i|.
\end{aligned} \tag{21}$$

This expression is a weighted version of $\boldsymbol{C}_\mathrm{c}(t)$, with weights given by the diagonal elements of ϱ_c. It may be understood as a mapping of the density matrix ϱ_c onto a diagonal matrix. It will be denoted by

$$\boldsymbol{\mathcal{C}}_\mathrm{c}(t)\ :\quad \varrho_\mathrm{c}\ \mapsto\ \boldsymbol{\mathcal{C}}_\mathrm{c}[\varrho_\mathrm{c}](t) = \sum_{ik} |i\rangle\, \mathrm{e}^{-\mathrm{i}(E_k-E_i)t}\ \varrho^\mathrm{c}_{kk}\ \langle i|. \tag{22}$$

We shall find a generalization of $\boldsymbol{\mathcal{C}}_\mathrm{c}(t)$ again in the following section. Note that

$$\mathrm{tr}\,\boldsymbol{\mathcal{C}}_\mathrm{c}[\varrho_\mathrm{c}](t) = C_\mathrm{c}(-t). \tag{23}$$

We may write:

$$\langle A_I\rangle = \int \int_0^t \mathrm{d}\tau \mathrm{d}\tau'\, C_\mathrm{e}(\tau'-\tau)\, \boldsymbol{C}_\mathrm{c}[\varrho_\mathrm{c}](\tau-\tau'). \tag{24}$$

The double integral $\langle A_I\rangle$ is Hermitian though the integrand is not. We thus have obtained the relevant quantities for computing the average density matrix in terms of correlation functions. In order to obtain an explicit expression, we can subsequently average over the environmental Hamiltonian.

2.2. Master equation

To obtain a master equation we will have to consider conditions which are not of central interest to this paper, but it is important to see that Markovian assumptions will lead to a master equation in a similar way as for other models. This approximation will only hold as long as times are much shorter than the Heisenberg time of the environment.

In order to arrive at the desired master equation, firstly, we assume that the timescale for decoherence is much shorter than the Heisenberg time τ_H of the environment. We then approximate the spectral correlation function in the environment with a delta function:

$$C_\mathrm{e}(t) \to \delta(t/\tau_\mathrm{H}). \tag{25}$$

In the case of the random matrix environment $C_e(t) \to \bar{C}_e(t)$, the latter also contains a smooth part, which becomes important at times of the order of τ_H and beyond; for the precise form of $\bar{C}_e(t)$ see [30]. Secondly, we assume that the state $\varrho(t)$ of the full system (central system and environment) remains a product state throughout the whole evolution. One common argument is that ϱ_e does not change in the course of time because the environment is much bigger than the central system [33]. Together, these approximations are consistent with a Markov approximation. We then use the linear response result to advance the density matrix $\langle \varrho_c(t) \rangle$ by one time step, which should be short compared to the decoherence time but large compared to the mixing time in the environment. We thereby obtain a closed differential equation for $\langle \varrho_c(t) \rangle$, the desired master equation.

As before, we restrict the discussion to the GUE coupling. Note though that a GOE coupling can be treated along the same lines and would yield precisely the same result. In this respect, on the level of a master equation approach, the evolution of the central system does not distinguish between systems where the coupling breaks the time-reversal symmetry and others where it does not. With equation (25) and due to $C_c(0) = m\, \mathbb{1}_c$, we find:

$$\begin{aligned} \langle A_J \rangle &= \int_0^t \mathrm{d}\tau \int_0^\tau \mathrm{d}\tau'\, \delta\left(\frac{\tau - \tau'}{\tau_H}\right) \left[\boldsymbol{C}_c(\tau - \tau')\, \varrho_c + \text{herm. conjg.} \right] \\ &= \frac{\tau_H^2}{2} \int_0^{t/\tau_H} \mathrm{d}x \left[\boldsymbol{C}_c(0)\, \varrho_c + \text{herm. conjg.} \right] = m\, t\, \tau_H\, \varrho_c, \end{aligned} \tag{26}$$

$$\langle A_I \rangle = \int_0^t \mathrm{d}\tau \int_0^t \mathrm{d}\tau'\, \delta\left(\frac{\tau - \tau'}{\tau_H}\right) \boldsymbol{\mathcal{C}}_c[\varrho_c](\tau - \tau') = \tau_H^2 \int_0^{t/\tau_H} \mathrm{d}x\, \mathbb{1}_c = \tau_H\, t\, \mathbb{1}_c, \tag{27}$$

where we have used the fact that $\boldsymbol{\mathcal{C}}_c[\varrho_c](0) = \mathbb{1}_c$. With these two results and (10), we obtain for the derivative of $\langle \tilde{\varrho}_c(t) \rangle$ at $t = 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \tilde{\varrho}_c(t) \rangle = -\lambda^2\, (m\, \tau_H\, \varrho_c - \tau_H\, \mathbb{1}_c\,). \tag{28}$$

Since we are working in the vicinity of $t = 0$, we may replace ϱ_c with $\langle \tilde{\varrho}_c(0) \rangle$ on the right-hand side and thereby close the above relation:

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \tilde{\varrho}_c(t) \rangle = -m\, \tau_H\, \lambda^2\, (\, \langle \tilde{\varrho}_c(t) \rangle - m^{-1}\, \mathbb{1}_c). \tag{29}$$

We therefore obtain as a closed evolution equation for the density matrix of the central system

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \varrho_c(t) \rangle = -\mathrm{i}\, [H_c,\, \langle \varrho_c(t) \rangle] - m\, \tau_H\, \lambda^2\, (\, \langle \varrho_c(t) \rangle - m^{-1}\, \mathbb{1}_c), \tag{30}$$

which constitutes the desired master equation [33] and establish that for fast decoherence we retrieve the known results.

2.3. Purity of the average density matrix

We calculate the purity on the basis of (15). That implies the calculation of two terms, $\mathrm{tr}(\langle A_J \rangle \varrho_c)$ and $\mathrm{tr}(\langle A_I \rangle \varrho_c)$. For the first term we obtain from (19):

$$\begin{aligned} \mathrm{tr}(\langle A_J \rangle \varrho_c) &= \int_0^t \mathrm{d}\tau \int_0^\tau \mathrm{d}\tau'\, \mathrm{tr}\big[\, C_e(\tau - \tau')\, \boldsymbol{C}_c(\tau - \tau')\, \varrho_c^2 + \text{herm. conjg.} \,\big] \\ &= \int \int_0^t \mathrm{d}\tau\, \mathrm{d}\tau'\, C_e(\tau - \tau')\, \mathrm{tr}\big[\, \boldsymbol{C}_c(\tau - \tau')\, \varrho_c^2 \,\big]. \end{aligned} \tag{31}$$

If ϱ_c is a pure state, $\varrho_c^2 = \varrho_c$ and

$$\mathrm{tr}(\langle A_J\rangle \varrho_c) = \int\int_0^t \mathrm{d}\tau \mathrm{d}\tau'\, C_e(\tau-\tau')\, C_c(\tau-\tau'). \tag{32}$$

Similarly, we obtain:

$$\begin{aligned}\mathrm{tr}(\langle A_I\rangle \varrho_c) &= \int\int_0^t \mathrm{d}\tau \mathrm{d}\tau'\, C_e(\tau-\tau')\, \mathrm{tr}\big[\, \boldsymbol{C}_c[\varrho_c](\tau-\tau')\, \varrho_c\,\big] \\ &= \int\int_0^t \mathrm{d}\tau \mathrm{d}\tau'\, C_e(\tau'-\tau)\, S_c(\tau-\tau'),\end{aligned} \tag{33}$$

where

$$S_c(t) = \sum_{ik} \mathrm{e}^{-\mathrm{i}(E_k-E_i)t}\, \varrho^c_{kk}\, \varrho^c_{ii}. \tag{34}$$

If ϱ_c is a pure state, the quantity $S_c(\tau-\tau')$ agrees with that defined in [30]. Overall, we obtain

$$P(t) \approx 1 - 2\lambda^2 \int_0^t \mathrm{d}\tau \int_0^t \mathrm{d}\tau' \left[\, C_e(\tau-\tau')\, C_c(\tau-\tau') - C_e(\tau'-\tau)\, S_c(\tau-\tau')\,\right]. \tag{35}$$

For a large environment ($N \to \infty$) and for a pure state ϱ_c, this result agrees with the average purity calculated in [30]. For finite N, their difference is calculated in appendix B. Note that neither result requires any averaging over the dynamics in the environment. Averaging over the coupling is sufficient.

2.4. Towards exact expressions for the averaged density matrix

For small dimensions of the central system exact expressions for the ensemble averaged entries of the reduced density matrix can in principle be obtained by methods similar to the ones used for the calculation of fidelity decay [28, 29, 34]. For simplicity let's assume the central system to be a two-level system (i.e. one qubit), initially pure. We write

$$\rho_c(t) = \sum_{n=0}^{3} \rho_{cn}(t)\sigma_n, \tag{36}$$

where σ_i, $i = 1, 2, 3$ are Pauli-matrices and σ_0 is the unit matrix. Then the ensemble average of $\rho_{cn}(t)$ takes the form

$$\overline{\rho_{cn}(t)} = \mathrm{tr}\left[\overline{\mathrm{e}^{-\mathrm{i}H_\lambda t}(\rho_c\otimes\rho_e)\,\mathrm{e}^{\mathrm{i}H_\lambda t}(\sigma_n\otimes 1_N)}\right]. \tag{37}$$

We use an over-line to denote the ensemble average, because here, the average is not only over the coupling but also over the environmental Hamiltonian and even over initial states. Due to the latter, the above expression further simplifies to

$$\overline{\rho_n(t)} = \sum_{m=0}^{3} \rho_{cm}(0)\, f_{nm}(t), \qquad f_{nm}(t) = \frac{1}{2N}\,\mathrm{tr}\left[\overline{\mathrm{e}^{-\mathrm{i}H_\lambda t}\,\mathrm{e}^{\mathrm{i}\sigma_m H_\lambda \sigma_n t}}\right]. \tag{38}$$

The quantities $f_{nm}(t)$ are formally average fidelity amplitudes or—equivalently—the ensemble averages of the trace of echo operators. The forward time evolution is given by the $2N \times 2N$ matrix $\sigma_m H_\lambda \sigma_n$ and the backward time evolution is propagated by H_λ. Fidelity amplitudes have been calculated exactly for several different random matrix ensembles using supersymmetry.

The calculation of $f_{nm}(t)$ is therefore amenable to an exact treatment using the techniques of [10, 26, 29, 35]. We shall present the result in a future paper.

At this point, we can address a question that is of interest[7]. In our model, we implicitly assume a random state in the environment, and on average constant couplings between all states. The first aspect implies a high temperature approximation for the environment. Yet we would also expect couplings to states with very different energy to be small. In terms of a random matrix model, this would imply the use of banded matrices rather than full matrices for the coupling. If we think in terms of linear response or any perturbation approximation, the large energy denominators will correct for the fact that the matrix is not banded. In the framework of such approximations, we can thus expect that we do not incur a big error. If we actually use an eigenstate of the entire system then the situation is different. Such a situation has been discussed for the RMT models of fidelity [21] and a significant effect was seen, which diminished in terms of an increasing participation ratio of the uncoupled states in terms of the coupled ones. Small participation ratios can only be expected for weak couplings combined with very low level densities of the environment. Indeed the large N approximation to obtain explicit formulae would not hold and we must make calculations similar to the ones in the above reference. Yet the question arises whether there would be a non-perturbative effect in the exact solution, which is not realistic. While we have no strict answer at this point, the fact that Lutz and Weidenmueller get a similar result from a banded random coupling in the case of fast decay is reassuring, as it is for short times that we expect to see effects of far away energies.

3. Decoherence within the spectator model

Dynamical model. In the spectator model, the central system is divided into two parts $\mathcal{H}_c = \mathcal{H}_1 \otimes \mathcal{H}_2$, where only one part is coupled to the environment (in our case this will be $\mathcal{H}_1$). The Hilbert space of the environment is as before $\mathcal{H}_e$. The dimension of the different spaces are: $m_1 = \dim(\mathcal{H}_1)$, $m_2 = \dim(\mathcal{H}_2)$ and $N = \dim(\mathcal{H}_e)$. The dimension of the total Hilbert space is $N\,m$, where $m = m_1\,m_2$. The Hamiltonian reads

$$H = H_1 + H_2 + H_e + V_{1,e} \otimes \mathbb{1}_2. \tag{39}$$

For an initial state $\varrho_c \otimes \varrho_e$ this means only $\mathcal{H}_1$ and $\mathcal{H}_e$ are coupled dynamically, whereas $\mathcal{H}_2$ may affect the dynamics of the whole system only due to an initial entanglement, i.e. a non-separable ϱ_c.

Linear response approximation. The spectator model may be treated along the same lines as the non-structured model considered in section 2. We may use equations (10) and (12) without change, and the principal task is to compute $\langle A_J \rangle$ and $\langle A_I \rangle$ on the basis of (16) and (20), respectively. We only have to calculate the coupling matrix in the interaction picture together with its second moment anew. We introduce $u_1(t) = \exp(-\mathrm{i}H_1 t)$, $u_2(t) = \exp(-\mathrm{i}H_2 t)$ and denote by $V_{1,e}$ the coupling matrix acting on the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_e$ and obtain:

$$\begin{aligned}\tilde{V}_{1,e}(t) &= u_1(t)^\dagger \otimes u_e(t)^\dagger \otimes u_2(t)^\dagger \left[V_{1,e} \otimes \mathbb{1}_2 \right] u_1(t) \otimes u_e(t) \otimes u_2(t) \\ &= \left[u_1(t)^\dagger \otimes u_e(t)^\dagger \, V_{1,e} \, u_1(t) \otimes u_e(t) \right] \otimes \mathbb{1}_2. \end{aligned} \tag{40}$$

[7] We thank the referee for pointing this out.

This result can be interpreted as the 'old' random coupling matrix in the interaction picture acting in the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_e$ times the identity in $\mathcal{H}_2$. As a consequence:

$$\langle \tilde{V}_{1,e}(\tau)\, \tilde{V}_{1,e}(\tau')\rangle = \boldsymbol{C}_1(\tau-\tau') \otimes \mathbb{1}_2 \otimes \boldsymbol{C}_e(\tau-\tau'), \tag{41}$$

$$\langle A_J\rangle = \int_0^t \mathrm{d}\tau \int_0^\tau \mathrm{d}\tau' \left[C_e(\tau-\tau')\, \boldsymbol{C}_1(\tau-\tau') \otimes \mathbb{1}_2\, \varrho_c + \text{herm. conjg.}\right]. \tag{42}$$

In the case of $\langle A_I\rangle$ the calculation is a bit more involved:

$$\begin{aligned}
\mathrm{tr}_e\left[\langle \tilde{V}_{1,e}(\tau)\, \varrho_0\, \tilde{V}_{1,e}(\tau')\rangle\right] &= |ij\rangle\, \mathrm{e}^{\mathrm{i}(E_i+E_j+E_\alpha)\tau} \Big\langle V^{1,e}_{i\alpha,k\beta}\, \delta_{jl}\, \mathrm{e}^{-\mathrm{i}(E_k+E_l+E_\beta)\tau}\, \varrho^c_{kl,mn}\, \varrho^e_{\beta\gamma} \\
&\quad \times \mathrm{e}^{\mathrm{i}(E_m+E_n+E_\gamma)\tau'}\, V^{1,e}_{m\gamma,p\alpha} \Big\rangle \delta_{nq}\, \mathrm{e}^{-\mathrm{i}(E_p+E_q+E_\alpha)\tau'}\, \langle pq| \\
&= |ij\rangle\, \mathrm{e}^{-\mathrm{i}(E_k-E_i)(\tau-\tau')}\, \varrho^c_{kj,kn}\, \mathrm{e}^{-\mathrm{i}(E_\beta-E_\alpha)(\tau-\tau')}\, \varrho^e_{\beta\beta}\, \langle in| \\
&= C_e(\tau'-\tau) \sum_{ij,kn} |ij\rangle\, \mathrm{e}^{-\mathrm{i}(E_k-E_i)(\tau-\tau')}\, \varrho^c_{kj,kn}\, \langle in|.
\end{aligned} \tag{43}$$

This result is quite similar to the expression in (21) for the same quantity in the non-structured case, where we had defined the operator-mapping $\boldsymbol{\mathcal{C}}_c(t)$. One prominent feature of this mapping was the fact that it transformed any matrix into a diagonal one. In the present expression this occurs only partially, namely in $\mathcal{H}_1$, whereas the density matrix remains in a sense untouched in $\mathcal{H}_2$. Thus we generalize $\boldsymbol{\mathcal{C}}_c(t)$, in what follows $\boldsymbol{\mathcal{C}}_1(t)$, to cases where its argument acts on a product space $\mathcal{H}_c = \mathcal{H}_1 \otimes \mathcal{H}_2$, in which case the operator mapping yields exactly what we obtained in equation (43).

$$\boldsymbol{\mathcal{C}}_1(t)\ :\quad \varrho_c \mapsto \boldsymbol{\mathcal{C}}_1[\varrho_c](t) = \sum_{ij,kn} |ij\rangle\, \mathrm{e}^{-\mathrm{i}(E_k-E_i)(\tau-\tau')}\, \varrho^c_{kj,kn}\, \langle in|. \tag{44}$$

This allows us to write:

$$\langle A_I\rangle = \int \int_0^t \mathrm{d}\tau\, \mathrm{d}\tau'\, C_e(\tau'-\tau)\, \boldsymbol{\mathcal{C}}_1[\varrho_c](\tau-\tau'). \tag{45}$$

3.1. Master equation

Following the same lines as the derivation of the master equation in the non-structured case, we find in the Fermi golden rule limit:

$$\langle A_J\rangle = m_1\, \tau_H\, t\, \varrho_c, \qquad \langle A_I\rangle = \tau_H\, t\, \mathbb{1}_1 \otimes \mathrm{tr}_1\, \varrho_c. \tag{46}$$

Therefore

$$\langle \tilde{\varrho}_c(t)\rangle = \varrho_c - \lambda^2\, \tau_H\, t \left[m_1\, \varrho_c - \mathbb{1}_1 \otimes \mathrm{tr}_1\, \varrho_c \right], \tag{47}$$

which may be continued in time, just as in the non-structured case, to give a closed differential equation for $\langle \tilde{\varrho}_c(t)\rangle$:

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \tilde{\varrho}_c(t)\rangle = -m_1\, \tau_H\, \lambda^2 \left[\langle \tilde{\varrho}_c(t)\rangle - m_1^{-1}\, \mathbb{1}_1 \otimes \mathrm{tr}_1\, \langle \tilde{\varrho}_c(t)\rangle \right], \tag{48}$$

where $\langle \tilde{\varrho}_c(0)\rangle = \varrho_c$. This again shows that we retrieve standard behaviour for fast decoherence.

Werner state solution. Recently, [25, 30], it has been observed that the average purity and the average concurrence share a one-to-one correspondence during the decay of an initial Bell state. This correspondence is the same as for the one-parameter family of Werner states. Among other things, [25, 30] consider the decoherence in a central two-qubit system coupled in the spectator configuration to a random matrix environment, and it is under these circumstances that the curious relationship can be observed. The master equation derived above resolves this puzzle at least for fast decoherence, since Werner states are solutions to that equation, as we shall show below. The average purity (concurrence) agrees with the purity (concurrence) of the average density matrix in the limit of a large environment ($N \to \infty$). In the case of purity, this has been shown analytically within the linear response approximation and numerically in section 4. In the case of concurrence this has been shown only numerically (in the same section 4).

Let us consider the two-qubit system of section 4. We define a Werner state as a two-qubit mixed state which may be decomposed into a maximally entangled pure state and the identity matrix:

$$\varrho_{\mathrm{W}}(t) = \alpha(t)\,\frac{\mathbb{1}_{\mathrm{c}}}{4} + \beta(t)\,|\psi\rangle\langle\psi|, \tag{49}$$

where ψ is the maximally entangled initial state, and $\alpha(t) + \beta(t) = 1$. Maximal entanglement is equivalent to the condition

$$\mathrm{tr}_1\,|\psi\rangle\langle\psi| = \frac{\mathbb{1}_2}{2} \quad \Leftrightarrow \quad \mathrm{tr}_2\,|\psi\rangle\langle\psi| = \frac{\mathbb{1}_1}{2}. \tag{50}$$

We will show that $\varrho_{\mathrm{W}}(t)$ solves the master equation (48) for appropriately chosen coefficients $\alpha(t)$ and $\beta(t)$. Note that (48) determines the time evolution of the two-qubits in the interaction picture. In order to obtain $\langle\varrho_{\mathrm{c}}(t)\rangle$ which describes the time evolution of the central system in the Schrödinger picture, we have to apply (12). If we have some single qubit dynamics, that may result in $\langle\varrho_{\mathrm{c}}(t)\rangle$ being no longer a Werner state. To complete the proof, we simply make the ansatz:

$$\tilde{\varrho}_{\mathrm{c}}(t) = \varrho_{\mathrm{W}}(t), \quad \tilde{\varrho}_{\mathrm{c}}(0) = |\psi\rangle\langle\psi|. \tag{51}$$

Substitution into the master equation (48) yields:

$$\dot{\alpha}\,\frac{\mathbb{1}_{\mathrm{c}}}{4} + \dot{\beta}\,|\psi\rangle\langle\psi| = -2\tau_{\mathrm{H}}\,\lambda^2\,\beta\left(|\psi\rangle\langle\psi| - \frac{\mathbb{1}_{\mathrm{c}}}{4}\right). \tag{52}$$

Taking the trace on either side, we find: $\dot{\alpha} + \dot{\beta} = 0$, in agreement with the normalization condition above. However, projecting either side on $|\psi\rangle\langle\psi|$, we obtain:

$$\frac{\dot{\alpha}}{4} + \dot{\beta} = -2\tau_{\mathrm{H}}\,\lambda^2\,\frac{3\beta}{4} \quad \Rightarrow \quad \dot{\beta} = -2\tau_{\mathrm{H}}\,\lambda^2\,\beta \quad \Rightarrow \quad \beta(t) = \mathrm{e}^{-2\tau_{\mathrm{H}}\,\lambda^2\,t}. \tag{53}$$

As announced, this determines $\alpha(t)$ and $\beta(t)$ in such a way that the Werner state $\varrho_{\mathrm{W}}(t)$ solves the master equation.

3.2. Purity

The linear response expression (15) for purity equally holds for the present spectator model. Thus

$$P(t) = \mathrm{tr}\,\langle\tilde{\varrho}_{\mathrm{c}}(t)\rangle^2 \approx \mathrm{tr}\varrho_{\mathrm{c}}^2 - 2\lambda^2\,\mathrm{tr}\big[\,\mathrm{tr}(\langle A_J\rangle\varrho_{\mathrm{c}}) - \mathrm{tr}(\langle A_I\rangle\varrho_{\mathrm{c}})\,\big], \tag{54}$$

where

$$\mathrm{tr}(\langle A_J\rangle \varrho_\mathrm{c}) = \int\int_0^t \mathrm{d}\tau \mathrm{d}\tau'\, C_\mathrm{e}(\tau-\tau')\, \mathrm{tr}\big[\, \boldsymbol{C}_1(\tau-\tau')\otimes \mathbb{1}_2\, \varrho_\mathrm{c}^2\,\big] \tag{55}$$

$$\mathrm{tr}(\langle A_I\rangle \varrho_\mathrm{c}) = \int\int_0^t \mathrm{d}\tau \mathrm{d}\tau'\, C_\mathrm{e}(\tau'-\tau)\, \mathrm{tr}\big[\, \boldsymbol{C}_1[\varrho_\mathrm{c}](\tau-\tau')\, \varrho_\mathrm{c}\,\big]. \tag{56}$$

If ϱ_c represents a pure state, the purity $P(t)$ in linear response approximation depends on the initial state only via $\mathrm{tr}_2\, \varrho_\mathrm{c}$. This can be seen from:

$$\mathrm{tr}(\langle A_J\rangle \varrho_\mathrm{c}) = \int\int_0^t \mathrm{d}\tau \mathrm{d}\tau'\, C_\mathrm{e}(\tau-\tau')\, \mathrm{tr}\big[\, \boldsymbol{C}_1(\tau-\tau')\, \mathrm{tr}_2\, \varrho_\mathrm{c}^2\,\big], \tag{57}$$

which is true for any density matrix ϱ_c, and

$$\mathrm{tr}(\langle A_I\rangle \varrho_\mathrm{c}) = \int\int_0^t \mathrm{d}\tau \mathrm{d}\tau'\, C_\mathrm{e}(\tau'-\tau)\, \mathrm{tr}\big[\, \boldsymbol{C}_1[\mathrm{tr}_2\, \varrho_\mathrm{c}](\tau-\tau')\,\big], \tag{58}$$

which is only true if ϱ_c represents a pure state.

4. Two-qubit central system

We shall proceed to analyse properties of the average density matrix for the particular case of a central system consisting of two qubits. Most of the analysis will be numerical, and thus we need a small central system for expedience. A single qubit is of interest, but choosing this case would eliminate the option of considering entanglement within the central system. Anyway, the one qubit case will emerge as a special case, if the spectator is not entangled.

For the two qubit case, we consider not only the purity of the density matrix ρ but also its von Neumann entropy $S(\rho) = -\mathrm{tr}\rho\log\rho$. The latter quantity is preferred by many authors as a measure of decoherence. We shall quantify internal entanglement with concurrence, and study its properties in the spirit mentioned above. It is defined as $C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$ where λ_i are the eigenvalues of the matrix $\sqrt{\rho(\sigma_y\otimes\sigma_y)\rho^*(\sigma_y\otimes\sigma_y)}$ in non-increasing order, $(^*)$ denotes complex conjugation in the computational basis, and σ_y is a Pauli matrix [36].

In figure 1, we show the differences between the ensemble averages of C and S as well as their value for the average density matrix $\bar{\rho}$, for different times and stepwise increasing the size of the environment. As the reader can see this difference approaches zero with increasing environment size as $1/N_\mathrm{env}$. Similar results were obtained for purity, but are not shown in the figure. They are consistent with the results found in section 2. The upshot is that all these very relevant quantities do not depend on the form in which the average is taken. That this is a consequence of a concentration of measure to some generalized delta function may be conjectured, but the demonstration will be an interesting subject of future research.

Concurrence and purity, for two non-interacting qubits coupled moderately to an environment, display a relation which corresponds well to that of a Werner state, while the members of the ensemble are *not* Werner states. It is thus reasonable to ask whether the average density matrix corresponds to a Werner state. A two-qubit Werner state, can be alternatively defined as a density matrix with three degenerate eigenvalues, whose non-degenerate eigenstate is a maximally entangled state (i.e. a Bell state modulo local unitary transformations). We first examine under which circumstances we have a triple degeneracy.

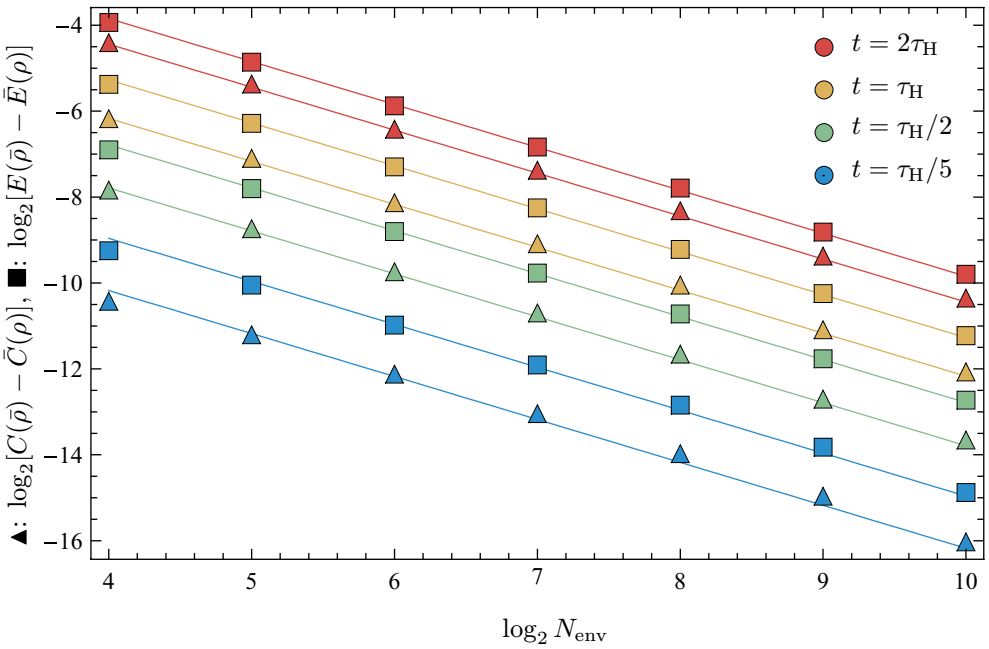

Figure 1. Difference between the entropy (squares)/concurrence (triangles) of the average density matrix and their value for each density matrix, averaged over the ensemble. The colours indicate different times and the lines correspond to a fit with fixed slope of -1. This shows that the average density matrix captures the physics of the ensemble for the quantities considered, including purity in the large N limit. Here, $\lambda = 0.03$, and the average values of purity are $\bar{P}(\tau_{\rm H}/5) \approx 0.99$, $\bar{P}(\tau_{\rm H}/2) \approx 0.95$, $\bar{P}(\tau_{\rm H}) \approx 0.88$ and $\bar{P}(2\tau_{\rm H}) = 0.66$.

Numerically, we construct a finite ensemble of $N_{\rm tot}$ density matrices. We partition it in sets of equal size $N_{\rm par}$, and evaluate the average density matrix $\rho^{(i)}_{N_{\rm par}}$ for each of the $N_{\rm tot}/N_{\rm par}$ members of the partition ($i = 1, \ldots, N_{\rm tot}/N_{\rm par}$). The properties of $\rho^{(i)}_{N_{\rm par}}$ are studied. In particular, we consider the standard deviation $\sigma_{\rm Werner}$ of its three closest eigenvalues. The average $\sigma_{\rm Werner}$ for each of the $N_{\rm tot}/N_{\rm par}$ sets available is plotted in figure 2 for different level splitting Δ, [$H_{\rm c} = \Delta\sigma_z/2$ in (3)].

Indeed, if the coupled qubit has no internal dynamics, the average state, in the limit of large dimension for the environment, is triply degenerate. In fact $\sigma_{\rm Werner} \to 0$ as $1/\sqrt{N_{\rm par}}$. If the qubit has an internal Hamiltonian, the degeneracy is lifted: a systematic splitting of the eigenvalues (and hence a deviation from Werner states) happens on a timescale set by the mean level spacing of the environment.

For the case of no internal dynamics, the fact that the average density matrix for two qubits is a Werner state if the initial state is a Bell state (or indeed a Werner state) can be readily proven at least for the spectator configuration. To show this we proceed in two steps. First, we consider a single qubit. Its density matrix can always be written as a pure state plus the identity with their respective weights. For a GUE random matrix ensemble the action on any initial state is entirely independent of this state due to the unitary invariance of this ensemble, i.e. it acts as a depolarizing channel. On the other hand, there being no internal Hamiltonian, the pure part has to remain unchanged. We can now proceed to calculate the effect on a Bell state of two

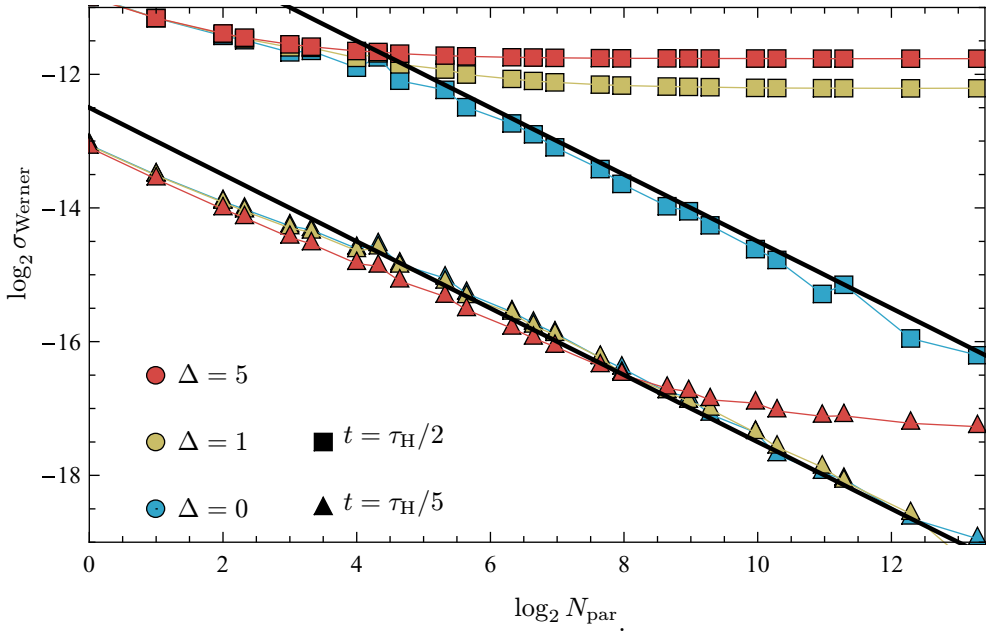

Figure 2. Standard deviation of the three closest eigenvalues of the average density matrix as a function of the size of the ensemble (see text for details). Different level splitting (differentiated by colours) and different times (differentiated by symbol type) are studied. When the coupled qubit has no internal Hamiltonian (blue symbols), the density matrix ρ has a triply degenerate eigenvalue in the large dimension limit. However when $\Delta > 0$, this triple degeneracy is lifted, as can be seen from an asymptotic nonzero value.

qubits. We parametrize the depolarizing channel $\mathcal{E}_{\text{dep}}$ in a single qubit with the Kraus operators $\{\sqrt{1-3p/4}\mathbb{1}, \sqrt{p}\sigma_x/2, \sqrt{p}\sigma_y/2, \sqrt{p}\sigma_z/2\}$ [37]. The action of the same channel acting on one of the qubits that form a Bell state, say the first qubit, can be calculated easily:

$$\begin{aligned}
\mathcal{E}_{\text{dep}} \otimes \mathbb{1}[|\text{Bell}\rangle\langle\text{Bell}|] &= \left(1 - \frac{3p}{4}\right)|\text{Bell}\rangle\langle\text{Bell}| + \sigma_x \otimes \mathbb{1}|\text{Bell}\rangle\langle\text{Bell}|\sigma_x \otimes \mathbb{1} \\
&\quad + \sigma_y \otimes \mathbb{1}|\text{Bell}\rangle\langle\text{Bell}|\sigma_y \otimes \mathbb{1} + \sigma_z \otimes \mathbb{1}|\text{Bell}\rangle\langle\text{Bell}|\sigma_z \otimes \mathbb{1} \\
&= \left(1 - \frac{3p}{4}\right)|\text{Bell}\rangle\langle\text{Bell}| + \frac{p}{4}(|01\rangle\langle 01| + |01\rangle\langle 01|) \\
&\quad + \frac{p}{8}(|00\rangle\langle 00| - |00\rangle\langle 11| - |11\rangle\langle 00| + |11\rangle\langle 11|) \\
&= p\frac{\mathbb{1}}{4} + (1-p)|\text{Bell}\rangle\langle\text{Bell}| \qquad (59)
\end{aligned}$$

with $|\text{Bell}\rangle = (|00\rangle + |11\rangle)/2$.

We have shown that for a spectator configuration the average density matrix resulting from applying a GUE environment and coupling to a Bell state will result in a Werner state independent for strong and weak couplings and short as well as long times. It trivially generalizes to the case where the initial state is a Werner state. This result is *independent*

of the dimension of the environment. Numerics show that it is also true or a very good approximation if we have a Werner state as initial state. Note that the case of time reversal invariant GOE dynamics is a little more complicated, as these ensembles are not invariant under all transformations of Hilbert space [38]. As we know that ensemble averaged quantities approach the results for the average density matrix in the limit of large environments this also explains the results in [20].

We also studied the concurrence of the eigenvector corresponding to the non-degenerate eigenvalue. It remains very close to one, independently of the concurrences of the other eigenvectors, which fluctuate without a clear pattern. We find the rather surprising result that this property of the eigenfunction of the largest eigenvalue persists even in cases where the previous test shows that we no longer deal with Werner states. Whether this property is characteristic of the average density matrix or whether it is common to many models will have to be analysed in future work.

5. Ensembles of density matrices and average density matrix

The very spirit of the treatment of decoherence and entanglement by RMT leads to ensembles of density matrices: each member of the RMT ensemble (and, in our approach, also each initial condition in the environment) induces a density matrix ρ_i via unitary dynamics followed by partial tracing. For that ensemble of density matrices we may consider the average density matrix $\bar{\rho} = \overline{\rho_i}$. A difference of background will lead at this point to a natural inclination to two different approaches: physicists with a background in decoherence, quantum optics and quantum information will most likely tend to use this average density matrix to calculate any quantity desired [say $C(\bar{\rho})$], whereas others with a background in RMT will be inclined to calculate the ensemble averages of the quantities of interest [say $\overline{C(\rho_i)}$]. The difference between these two approaches disappears when one studies expected values of observables as $\mathrm{tr} A\bar{\rho} = \overline{\mathrm{tr} A\rho_i}$ or, as shown in this paper, for some nonlinear quantities in the limit of large dimension for the environment. Moreover, with quantitative entanglement witnesses [39], one can bound some of these nonlinear quantities using (linear) observables. Yet the very concept of a relevant near environment, important for many of the new insights obtained from an RMT treatment, suggests considering small environments as well. This leads us to discuss the appropriateness of either approach in typical applications.

One must recall that the determination of any of these quantities is not achievable as a single measurement. Rather we need an ensemble of 'identical' systems. The hinge is what we mean by 'identical'. If we have no control over the state or dynamics of the near environment, we must consider $f(\bar{\rho})$ (f being an expectation value, or any of the nonlinear quantities discussed). Two notes are important at this point: firstly, the problem discussed here appears for any type of averaging, not only for averaging over time evolutions. Secondly, for the GUE a state average is implicit in the ensemble average, whereas this is not the case for a GOE [38], and thus has to be performed separately. However, if the near environment is known and its dynamics can be controlled, one should consider $\overline{f(\rho)}$. This can indeed be the case e.g. for an ion trap setup such as the one discussed in the introduction, which might be used to experimentally verify our results.

The differentiation between the two approaches is also relevant within quantum information tasks. Consider for instance Shor's algorithm. The success rate has to be calculated from the average density matrix $\bar{\rho}$ resulting from many realizations of the same experiment. For

the teleportation of an unknown state, however, it is important to know the success rate at each attempt, and thus the quality of the process should result from an average quality for individual density matrices corresponding to each try.

At this point, to avoid confusion, we should recall that there is the option of obtaining a mixed density matrix without any entanglement by allowing probabilistic variations in the unitary time evolution of the central system itself in the absence of any significant coupling to the environment. Clearly we then have an ensemble of pure states, whose average naturally will not be pure. Yet errors due to variations of the unitary evolution are usually considered as a loss of fidelity. The purity of the mixed density matrix would then measure an average quantity closely related to fidelity. This explains our emphasis throughout this paper on the fact that we need a fixed evolution for the central system. It also shows that the separation of external and internal perturbations for the study of the stability of quantum information tasks is somewhat artificial in a practical sense, though it is very useful for theoretical studies.

The upshot of this discussion resides in the very fact that for many quantum information tasks, we need repeated experiments. The decision of which average should be taken is a subtle one. This emphasizes the importance of the result, indicating that the averages coincide in many cases.

6. Conclusions

We have presented a RMT of decoherence and have applied this model to study some results about the ensuing decay of entanglement in a bipartite central system. Intuitive arguments based on Fermi's golden rule allow us to derive the usual exponential decay of coherence almost independently of the model used for environment and coupling. The basic condition is that decoherence occurs long before this the Heisenberg time of the environment. Yet we argued that situations with a near environment with fairly low level density and thus short Heisenberg times occur. Indeed they will become standard as quantum information systems with even better isolation from the general environment are developed, and this development is essential to the practical success of quantum information. It is under these circumstances that RMT can provide the generic model to which the behaviour of specific systems should be compared. It will allow to distinguish purely statistical effects from specific ones, and serve to estimate the former.

After describing the family of RMT models, proposed to a large extent in earlier work, we proceed to analyse a point which has set RMT models apart from other models of decoherence. The ensemble of evolution operators creates for the central system an ensemble of density matrices rather than a single density matrix. Consequently, properties such as entropy, purity and concurrence have so far been calculated as averages over that ensemble. However, we can also compute the average density matrix first, putting us on equal footing with other more conventional models. The quantities mentioned above are then determined from that single density matrix. This has the great additional advantage of producing lower order quantities that have a fair chance of being calculated exactly using super-symmetric techniques. While we have not yet achieved this goal, we have calculated the average density matrix in linear response approximation. This as well as numerics allowed us to compare purity, von Neumann entropy as well as concurrence of the average density matrix to the average of these quantities over the ensemble of density matrices. The central finding is that for large environments at constant Heisenberg time (or mean level distance) the difference between the two approaches converges to zero as the inverse of the dimension of the relevant Hilbert space of the environment. This

indicates that for decoherence we can often use the average density matrix. This puts the RMT models on a similar footing to usual descriptions. Indeed the average density matrix will obey some evolution that is not simply obtained by a unitary evolution traced over the environment. Yet this evolution is obtained by ensemble averaging from unitary quantum evolution of the entire system in each member of the ensemble. The ensemble average is the only crucial deviation from unitary evolution, and does not imply any uncontrolled approximation.

Analysing the special case of two-qubit central systems numerically, the general results of our analysis were confirmed. We confirmed that in the interaction picture the average density matrix decays to good approximation along the family of Werner states. We also found the more surprising fact that at least for small decoherence, even in the Schrödinger picture, the eigenfunction of the dominant eigenvalue of the average density matrix remains a very good approximation of a Bell state, if the initial state was a Bell state.

Yet we have to note that, when describing the entanglement between two smaller systems, deviations between ensemble averaged quantities and the corresponding quantities calculated from average density matrices are important and the question of which average the behaviour of a single system should be compared to depends on the particular experimental situation. Nevertheless we have shown that purity and other quantities measuring entanglement yield the same result in the large environment limit, whether calculated from the average density matrix or as an average over the ensemble of density matrices. This was done analytically for purity in the linear response regime (section 2) and numerically, for purity, von Neumann entropy and concurrence beyond the linear response regime (section 4).

Taking into consideration the convexity of these quantities, this might imply that in the large dimension limit of the environment the measure for the density matrices becomes similar to a Dirac delta in the sense that all or a large class of convex functions could be calculated directly from the average density matrix. However, other results [20] suggest that typical states are far from the average expected state. This apparent contradiction as well as the distribution of the density matrices as such shall be studied in a later paper. This work can also be readily extended considering more realistic RMT ensembles. In particular the two-body random ensembles may play an important role particularly in their recent formulation for distinguishable spins [40]. Also, maps as generic models for gates are of possible interest, and we hope that we have laid the foundation for more research in the domain of decoherence by 'small' environments, where the Heisenberg time is shorter than or of the order of the decoherence time.

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Appendix A. Elements of RMT

In RMT, we start with an ensemble of Hamiltonians (or unitary time-evolution operators), which is defined by an ensemble of random matrices, where each member describes a different dynamics. In practice, the Hamiltonian matrices of a finite dimension, denoted by N, are used,

though often the limit of $N \to \infty$ is taken for analytic results. For an extensive introduction to the field, see e.g. [32].

In the present work, we use the GUE or the GOE. In both cases, the matrix elements are random Gaussian variables—complex in the GUE case, and real in the GOE case. Except for the requirements of symmetry, the matrix elements are statistically independent and have zero mean. Their variance is given by

$$\langle H_{ij}\, H_{kl}\rangle = \delta_{il}\,\delta_{jk} \tag{A.1}$$

in the GUE case, and

$$\langle H_{ij}\, H_{kl}\rangle = \delta_{il}\,\delta_{jk} + \delta_{ik}\,\delta_{jl} \tag{A.2}$$

in the GOE case. These ensembles have the advantage that they are invariant under arbitrary unitary (orthogonal) transformations. This means that whatever basis one chooses, the ensemble of Hamiltonian matrices will always look the same in the unitary case; in the orthogonal case only real states are equivalent.

The evolution operator corresponding to a random Hamiltonian matrix is given by

$$U(t) = \mathrm{e}^{-\mathrm{i}\,Ht/\hbar}. \tag{A.3}$$

Due to the unitary or orthogonal invariance of the ensemble, any initial state will lead to essentially the same kind of evolution. After an extremely short timescale (the Ehrenfest timescale, t_{Ehr}), any initial wave packet will be spread over the full Hilbert space [10]. For the Gaussian ensembles, t_{Ehr} only depends on the length of the spectrum, which is of the order of $\sqrt{N}$. A more realistic model would take into account that the Hamiltonian is typically banded and/or sparse, and this would lead to a somewhat larger Ehrenfest time. On the other hand, the Heisenberg time $t_{\mathrm{H}} = 2\pi\hbar/d$ (here, d is the average level spacing) is of the order of $\sqrt{N}$. For the Gaussian random ensembles, one finds that $t_{\mathrm{Ehr}} = t_{\mathrm{H}}/N$. Since these are the only two timescales available, their ratio is the only independent parameter. In our simulations, we consider the Heisenberg time to be fixed, and let the Ehrenfest time go to zero.

In the present work, we relate behaviour of the coherence of the central system as a function of time, to the two-point form factor [32] of the environment Hamiltonian. The two-point form factor is a statistical measure, which is defined in terms of $U(t)$. We call

$$|\langle\Psi_0|U(t)\,\Psi_0\rangle|^2 = \sum_{ij}\langle\Psi_0|j\rangle\,\mathrm{e}^{-\mathrm{i}E_j t/\hbar}\,\langle j|\Psi_0\rangle\,\langle\Psi_0|k\rangle\,\mathrm{e}^{\mathrm{i}E_k t/\hbar}\,\langle k|\Psi_0\rangle \tag{A.4}$$

the return probability for an initial state Ψ_0. Then, we find for the return probability averaged over the random matrix ensemble:

$$\left\langle|\langle\Psi_0|U(t)\,\Psi_0\rangle|^2\right\rangle = \left\langle\sum_j|\langle j|\Psi_0\rangle|^2\right\rangle + \left\langle\sum_{j\neq k}|\langle j|\Psi_0\rangle|^2\,\langle k|\Psi_0\rangle|^2\,\mathrm{e}^{-\mathrm{i}(E_j-E_k)t/\hbar}\right\rangle, \tag{A.5}$$

where the outer angular brackets denote the ensemble average. For the two-point form factor only the non-diagonal part of the double summation is relevant, such that we can define:

$$\delta(\tau) - b_2(\tau) = \left\langle\sum_{j\neq k}|\langle j|\Psi_0\rangle|^2\,|\langle k|\Psi_0\rangle|^2\,\mathrm{e}^{-2\pi\mathrm{i}(E_j-E_k)\,\tau/d}\right\rangle, \tag{A.6}$$

where the differences in the eigenenergies $E_j - E_k$ of the Hamiltonian are divided by the mean level spacing d, while time is measured in units of the Heisenberg time: $\tau = t/t_{\mathrm{H}}$.

In both the GUE and the GOE case, the two-point form factor is a symmetric function. For $\tau > 0$ it reads for the GUE case as

$$b_2(\tau) = \theta(\tau - 1)\,(1 - \tau), \tag{A.7}$$

and for the GOE case as

$$b_2(\tau) = 1 - 2t + t\ln(2t+1) - \theta(t-1)\left[\,(2 - 2t + \ln(2t-1)\,\right], \tag{A.8}$$

where $\theta(\tau)$ is the unit step function: $\theta(\tau) = 1$ for $\tau > 0$ and zero, otherwise.

Appendix B. Average purity for the non-structured and the spectator model

In this section, we review the results of [30] for the average purity instead of the purity of the average density matrix for the models treated in sections 2 and 3. We assume that the initial state is a product state of the form

$$\varrho_0 = \varrho_\mathrm{c} \otimes |\psi_\mathrm{e}\rangle\langle\psi_\mathrm{e}|, \tag{B.1}$$

where the state of the central system ϱ_c may be mixed, but the state of the environment $|\psi_\mathrm{e}\rangle\langle\psi_\mathrm{e}|$ must be pure. The latter is a special requirement which allowed to map the spectator model onto the non-structured model in [30]. For the average purity it was found that

$$\langle P(t)\rangle = P(0) - \lambda^2(\,B_J - B_I\,), \tag{B.2}$$

$$B_J = 4\,\mathrm{Re}\,p\big[\,\langle J(t)\rangle\,\varrho_0 \otimes \varrho_0\,\big], \tag{B.3}$$

$$B_I = 2\left(p\big[\,\langle I(t)\,\varrho_0\,I(t)\rangle \otimes \varrho_0\,\big] - \mathrm{Re}\,p\big[\,\langle I(t)\,\varrho_0 \otimes I(t)\,\varrho_0\,\big] + p\big[\,\langle I(t)\,\varrho_0 \otimes \varrho_0\,I(t)\,\big]\right), \tag{B.4}$$

$$p[A \otimes B] = \mathrm{tr_e}(\,\mathrm{tr_c}\,A\,\mathrm{tr_c}\,B\,). \tag{B.5}$$

With equation (17) we obtain

$$\begin{aligned} B_J &= 4\,\mathrm{Re}\int_0^t \mathrm{d}\tau \int_0^\tau \mathrm{d}\tau'\; p\big[\,\langle \tilde{V}(\tau)\,\tilde{V}(\tau')\rangle\,\varrho_0 \otimes \varrho_0\,\big] \\ &= 4\,\mathrm{Re}\int_0^t \mathrm{d}\tau \int_0^\tau \mathrm{d}\tau'\; p\big[\,(\,\boldsymbol{C}_\mathrm{c}(\tau-\tau') \otimes \boldsymbol{C}_\mathrm{e}(\tau-\tau')\,)\,\varrho_0 \otimes \varrho_0\,\big] \\ &= 4\,\mathrm{Re}\int_0^t \mathrm{d}\tau \int_0^\tau \mathrm{d}\tau'\;\mathrm{tr_e}(\,C_\mathrm{c}(\tau-\tau')\,\boldsymbol{C}_\mathrm{e}(\tau-\tau')\,\varrho_\mathrm{e}\,\varrho_\mathrm{e}\,). \end{aligned} \tag{B.6}$$

Since ϱ_e is a pure state: $\varrho_\mathrm{e}^2 = \varrho_\mathrm{e}$, and because $C_\mathrm{x}(\tau-\tau') = C_\mathrm{x}(\tau'-\tau)^*$, we obtain:

$$B_J = 4\,\mathrm{Re}\int_0^t \mathrm{d}\tau \int_0^\tau \mathrm{d}\tau'\; C_\mathrm{c}(\tau-\tau')\,C_\mathrm{e}(\tau-\tau') = 2\int\int_0^t \mathrm{d}\tau\mathrm{d}\tau'\; C_\mathrm{c}(\tau-\tau')\,C_\mathrm{e}(\tau-\tau'). \tag{B.7}$$

This is precisely the same quantity calculated in [30] whose final expression is given in equations (A.8) and (A.16). It shows that the absolute values taken in [30], though correct, are not really necessary.

To calculate the first term of B_I we note that

$$\begin{aligned} \langle \tilde{V}(\tau)\,\varrho_0\,\tilde{V}(\tau)\rangle &= |ij\rangle\,\mathrm{e}^{-\mathrm{i}(E_{kl}-E_{ij})(\tau-\tau')}\,\langle kl|\,\varrho_0\,|kl\rangle\,\langle ij| \\ &= \boldsymbol{\mathcal{C}}_\mathrm{c}[\varrho_\mathrm{c}](\tau-\tau') \otimes \boldsymbol{\mathcal{C}}_\mathrm{e}[\varrho_\mathrm{e}](\tau-\tau'). \end{aligned} \tag{B.8}$$

Therefore, with equation (23), which holds equally well in the case of the environment:

$$p\big[\,\langle I(t)\,\varrho_0\,I(t)\rangle\otimes\varrho_0\,\big] = \int_0^t\int_0^t \mathrm{d}\tau\mathrm{d}\tau'\,C_\mathrm{c}(\tau'-\tau)\,\mathrm{tr}_\mathrm{e}\big[\,\mathcal{C}_\mathrm{e}[\varrho_\mathrm{e}](\tau-\tau')\,\varrho_\mathrm{e}\,\big]$$
$$= \int_0^t\int_0^t \mathrm{d}\tau\mathrm{d}\tau'\,C_\mathrm{c}(\tau'-\tau)\,S_\mathrm{e}(\tau-\tau'), \tag{B.9}$$

where $S_\mathrm{e}(\tau-\tau')$ is defined in precise analogy to $S_\mathrm{c}(\tau-\tau')$ in equation (34). Similarly, we obtain for the following terms:

$$p\big[\,\langle I(t)\,\varrho_0\otimes I(t)\,\varrho_0\rangle\,\big] = \int_0^t\int_0^t \mathrm{d}\tau\mathrm{d}\tau'\,S_\mathrm{c}(\tau-\tau')\,S_\mathrm{e}(\tau-\tau'), \tag{B.10}$$

$$p\big[\,\langle I(t)\,\varrho_0\otimes\varrho_0\,I(t)\rangle\,\big] = \int_0^t\int_0^t \mathrm{d}\tau\mathrm{d}\tau'\,S_\mathrm{c}(\tau-\tau')\,C_\mathrm{e}(\tau'-\tau), \tag{B.11}$$

where we had to assume again that ϱ_c represents a pure state. In summary, we obtain:

$$\langle P(t)\rangle = 1-2\lambda^2\int_0^t\int_0^t \mathrm{d}\tau\mathrm{d}\tau'\,\big[\,C_\mathrm{e}(\tau-\tau')\,C_\mathrm{c}(\tau-\tau') - S_\mathrm{e}(\tau-\tau')\,C_\mathrm{c}(\tau'-\tau)$$
$$+\,S_\mathrm{e}(\tau-\tau')\,S_\mathrm{c}(\tau-\tau') - C_\mathrm{e}(\tau'-\tau)\,S_\mathrm{c}(\tau-\tau')\,\big], \tag{B.12}$$

where we have used the fact that the double integral over $S_\mathrm{e}(\tau-\tau')\,S_\mathrm{c}(\tau-\tau')$ is automatically real. The whole result is obviously invariant with respect to an interchange of the subscripts e and c. This means that we could have defined $p[A\otimes B]$ equally well by first tracing A and B over the environment and the resulting matrix product over the central system. This is equivalent to the statement that the purity of the state of the environment (after tracing over the central system) is equal to the purity of the state of the central system (after tracing over the environment).

If we compare the result for the average purity with equation (35) for the purity of the average density matrix, we find that the difference is equal to

$$\langle P(t)\rangle - P(t) = 2\lambda^2\int_0^t\int_0^t \mathrm{d}\tau\mathrm{d}\tau'\,S_\mathrm{e}(\tau-\tau')\,\big[\,S_\mathrm{c}(\tau-\tau') - C_\mathrm{c}(\tau'-\tau)\,\big]. \tag{B.13}$$

As discussed in [30], the function $S_\mathrm{e}(\tau)$ shows a similar behaviour to $C_\mathrm{e}(\tau)$, except for an additional factor of order N^{-1}. This holds at least if the state of the environment is sufficiently delocalized in energy. For that reason, we expect the difference $\langle P(t)\rangle - P(t)$ to be of order N^{-1}.

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