Decoherence of an n-Qubit Quantum Memory

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We analyze decoherence of a quantum register in the absence of nonlocal operations, i.e., n noninteracting qubits coupled to an environment. The problem is solved in terms of a sum rule which implies linear scaling in the number of qubits. Each term involves a single qubit and its entanglement with the remaining ones. Two conditions are essential: first, decoherence must be small, and second, the coupling of different qubits must be uncorrelated in the interaction picture. We apply the result to a random matrix model, and illustrate its reach considering a Greenberger-Horne-Zeilinger state coupled to a spin bath.

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Physical devices capable of storing faithfully a quantum state, i.e., quantum memories, are crucial for any quantum information task. While different types of systems have been considered, most of the effort has been concentrated in manipulating qubits as they are essential for most quantum information tasks [1]. For a general quantum memory (QM) it is necessary to record, store, and retrieve an arbitrary state. For quantum communication and quantum computation initialization in the base state, single qubit manipulations, a two qubit gate (e.g., control-not), and single qubit measurements are sufficient. For arbitrary quantum memories recording and retrieving the state is still an experimental challenge but initializing a qubit register and individual measurement has been mostly mastered. The faithful realization of two qubit gates together with better isolation from the environment are the remaining obstacles for achieving fully operational quantum technology. However, the huge effort made in the community has borne some fruit [2].

Understanding decoherence of one, two, and n qubits has been of interest for some time. Considering a bosonic thermal bath as the environment, some authors analyze the decoherence of a quantum register [3]. In some cases decoherence free subspaces were identified and in others the quantum properties were analyzed. The implications of the usual Markovian approximation are studied in [4]. In [5] Braun shows that a generalized Hamming distance influences the speed of decoherence. The effect of integrability and chaos in decoherence has been studied using spin baths [6,7]. Random Hamiltonians or parameters are also used [8,9]. In [10] the authors calculate decoherence of a QM for a spin-boson model using a measure designed for their purposes. In the limit of slight decoherence they obtain various results including additivity. Experimental results are also available [11]. One would wish to encompass some of this progress in a general picture.

In this Letter we address this problem using a standard measure of decoherence, namely, purity [12]. We obtain analytic expressions for the decoherence of a QM during the storage time. Specifically we discuss a QM composed of individual qubits interacting with some environment. The expressions given are based on previous knowledge of the decoherence of a single qubit entangled with some noninteracting spectator. Their validity is limited to small decoherence, i.e., large purity of the QM. Note that the latter is not a significant restriction due to the high fidelity requirements of quantum error correction codes. We assume that the entire system is subject to unitary time evolution, and that decoherence comes about by entanglement between the central system (CS) and the environment. Spurious interactions inside the central system are neglected. We shall rely heavily on our recent studies of decoherence of two qubits [8].

A further and critical assumption is the independence of the coupling of different qubits with the environment in the interaction picture. This is justified if the couplings are independent in the Schrödinger picture or if we have rapidly decaying correlations due to mixing properties of the environment [13]. Physically the first would be more likely if we talk about qubits realized in different systems or degrees of freedom, while the second seems plausible for many typical environments.

Our central result is a decomposition of the decoherence of the full QM, coupled to a single or several environments into a sum of terms. Each of these describes the decoherence of a single qubit in a "spectator configuration" [6,8] which is generalized as follows. The CS consists of two noninteracting parts, one (the qubit) interacting with the environment and the other (the rest of the QM) not. This configuration is nontrivial if the two parts of the CS are entangled. Apart from the above assumptions, this result does not depend on any particular property of the environment or coupling. Thus it can be applied to a variety of models.

We test successfully the results in a random matrix model for both coupling and environment [8]. The general relation is obtained in linear response approximation and leads to explicit analytic results if the spectral correlations of the environment are known. Finally, we perform numerical simulations for four qubits, interacting with a kicked Ising spin chain [6,14] as an environment, which can be chosen mixing or integrable.

The central system (our QM) is composed of n qubits. Thus, its Hilbert space is $\mathcal{H}_{\rm QM} = \otimes_{i=1}^n \mathcal{H}_i$, where $\mathcal{H}_i$ are the Hilbert spaces of the qubits. The Hilbert space of the environment is $\mathcal{H}_e$. The Hamiltonian reads as

$$H = H_{\rm QM} + H_e + \lambda V, \qquad \lambda V = \sum_{i=1}^n \lambda_i V^{(i)}. \tag{1}$$

Here $H_{\rm QM} = \sum_{i=1}^n H_i$, where H_i acts on $\mathcal{H}_i$, H_e describes the dynamics of the environment, and λV the coupling of the qubits to the environment. The strength of the coupling of qubit i is controlled by the parameter λ_i, while $\lambda = \max\{|\lambda_i|\}$. The (possibly time-dependent) Hamiltonian gives rise to the unitary evolution operator $U_\lambda(t)$.

We consider two different settings. In the first one $V^{(i)}$ acts on the space $\mathcal{H}_i \otimes \mathcal{H}_e$; i.e., all qubits interact with a single environment called *joint environment*. In the second one each qubit interacts with a *separate environment*. Thus the environment is split into n parts, $\mathcal{H}_e = \otimes_{i=1}^n \mathcal{H}_{e,i}$ and $V^{(i)}$, in Eq. (1), acts only on $\mathcal{H}_i \otimes \mathcal{H}_{e,i}$. The first case would be typical for a quantum computer, where all qubits are close to each other, while the second would apply to a nonlocal quantum network.

We choose the initial state to be the product of a pure state of the central system (the QM) and a pure state of the environment. It may be written as

$$|\psi_0\rangle = |\psi_{\rm QM}\rangle|\psi_e\rangle, \qquad |\psi_{\rm QM}\rangle \in \mathcal{H}_{\rm QM}, |\psi_e\rangle \in \mathcal{H}_e. \tag{2}$$

The reduced initial state in the QM, $\mathrm{Tr}_e|\psi_0\rangle\langle\psi_0| = |\psi_{\rm QM}\rangle\langle\psi_{\rm QM}|$, is pure. In the separate environment configuration we furthermore assume that $|\psi_e\rangle = \otimes_i|\psi_{e,i}\rangle$ with $|\psi_{e,i}\rangle \in \mathcal{H}_{e,i}$, which corresponds to the absence of quantum correlations among the different environments.

For reasons of analytic simplicity we chose the purity $P(t)$ as the measure for decoherence. It is defined as

$$P(t) = \mathrm{Tr}\rho_{\rm QM}^2(t), \qquad \rho_{\rm QM}(t) = \mathrm{Tr}_e U_\lambda(t)|\psi_0\rangle\langle\psi_0|U_\lambda^\dagger(t). \tag{3}$$

To calculate purity (or any other quantity that depends solely on the Schmidt coefficients) we can replace the forward time evolution $U_\lambda(t)$ by the echo operator $M(t) = U_0(t)U_\lambda(-t)$ where $U_0(t)$ gives the evolution without coupling. In linear response approximation we find

$$M(t) = \mathbb{1} - \imath\lambda I(t) - \lambda^2 J(t), \tag{4}$$

where $I(t) = \int_0^t d\tau \tilde{V}_\tau$ and $J(t) = \int_0^t d\tau \int_0^\tau d\tau' \tilde{V}_\tau \tilde{V}_{\tau'}$. $\lambda\tilde{V}_t = U_0(t)\lambda V U_0^\dagger(t)$ is the coupling at time t in the interaction picture ($\hbar \equiv 1$). As discussed in [8] it is convenient to introduce the form $p[\rho_1 \otimes \rho_2] = \mathrm{Tr}(\mathrm{Tr}_e\rho_1 \mathrm{Tr}_e\rho_2)$. Purity is then given by

$$\begin{aligned} P(t) &= 1 - 2\lambda^2 \int_0^t d\tau \int_0^t d\tau' \,\mathrm{Re}A(\tau,\tau') + O(\lambda^4),\\ A(\tau,\tau') &= p[\tilde{V}_\tau\tilde{V}_{\tau'}\varrho_0 \otimes \varrho_0] - p[\tilde{V}_{\tau'}\varrho_0\tilde{V}_\tau \otimes \varrho_0]\\ &\quad + p[\tilde{V}_\tau\varrho_0 \otimes \tilde{V}_{\tau'}\varrho_0] - p[\tilde{V}_{\tau'}\varrho_0 \otimes \varrho_0\tilde{V}_\tau], \end{aligned} \tag{5}$$

with $\varrho_0 = |\psi_0\rangle\langle\psi_0|$. Note that the linear terms vanish after tracing out the environment. Considering the form of the coupling in Eq. (1) we can decompose $\lambda^2 A(\tau,\tau') = \sum_{i,j}\lambda_i\lambda_j A^{(i,j)}(\tau,\tau')$ with

$$\begin{aligned} A^{(i,j)}(\tau,\tau') &= p[\tilde{V}_\tau^{(i)}\tilde{V}_{\tau'}^{(j)}\varrho_0 \otimes \varrho_0] - p[\tilde{V}_{\tau'}^{(i)}\varrho_0\tilde{V}_\tau^{(j)} \otimes \varrho_i]\\ &\quad + p[\tilde{V}_\tau^{(i)}\varrho_0 \otimes \tilde{V}_{\tau'}^{(j)}\varrho_0] - p[\tilde{V}_{\tau'}^{(i)}\varrho_0 \otimes \varrho_0\tilde{V}_\tau^{(j)}]. \end{aligned} \tag{6}$$

$\tilde{V}_t^{(i)} = U_0^\dagger(t)V^{(i)}U_0(t)$ in analogy with $\tilde{V}_t$. Equation (5) is given as a double sum in the indices of the qubits. In a diagonal approximation ($i = j$), $P(t)$ is expressed in terms of the purities $P_{\rm sp}^{(i)}(t)$ which correspond to the purity decay of the CS in a spectator configuration where only qubit i is interacting with the environment. Purity then reads as

$$\begin{aligned} P(t) &= 1 - \sum_{i=1}^n (1 - P_{\rm sp}^{(i)}(t)),\\ P_{\rm sp}^{(i)}(t) &= 1 - 2\lambda_i^2 \int_0^t d\tau \int_0^t d\tau' A^{(i,i)}(\tau,\tau') + O(\lambda_i^4). \end{aligned} \tag{7}$$

This is our central result. The diagonal approximation is justified in two situations: First, when the couplings $V^{(i)}$ of the individual qubits are independent from the outset, as would be typical for the separate environment configuration or for the random matrix model of decoherence [8]; second, when the couplings in the interaction picture become independent due to mixing properties of the environment, as would be typical for a "quantum chaotic" environment.

To illustrate the case of independent couplings mentioned above, random matrix theory provides a handy example. Such models were discussed in [8,15,16] and describe the couplings $V^{(i)}$ by independent random matrices, chosen from the classical ensembles [17].

In [8] the purity decay was computed in linear response approximation for two qubits, one of them being the spectator. For the sake of simplicity we choose the joint environment configuration, no internal dynamics for the qubits, and H_e and $V^{(1)}$ as typical members of the Gaussian unitary ensemble. Purity is then given by

$$P_{\rm sp}^{(1)}(t) = 1 - \lambda_1^2(2 - p_1)f(t); \tag{8}$$

$$f(t) = t\max\{t,\tau_H\} + \frac{2}{3\tau_H}\min\{t,\tau_H\}^3. \tag{9}$$

Here, τ_H is the Heisenberg time of the environment and p_1 is the *initial* purity of the first qubit alone, which measures its entanglement with the rest of the QM. As we required independence of the couplings, we can now insert Eq. (8)

into Eq. (7) to obtain the simple expression

$$P(t) = 1 - f(t)\sum_{i=1}^{n} \lambda_i^2 (2 - p_i), \tag{10}$$

where p_i is the initial purity of qubit i. In the presence of internal dynamics the spectator result is also known [8] and can be inserted. As an example, we apply the above equation to an initial Greenberger-Horne-Zeilinger (GHZ) state $[(|0\cdots 0\rangle + |1\cdots 1\rangle)/\sqrt{2}]$. Then all $p_i = 1/2$ and we obtain $P(t) = 1 - (3/2)f(t)\sum_i \lambda_i^2$. For a W state $(|10\ldots 0\rangle + |01\ldots 0\rangle + \cdots + |00\ldots 1\rangle)/\sqrt{n}$, purity for each qubit is $p_i = (n^2 - 2n + 2)/n^2 \approx 1$, in the large n limit, and purity decays as $P(t) = 1 - f(t)\sum_i \lambda_i^2$. For $n = 2$ we retrieve the results published in [8].

The main assumption in Eq. (7) is the fast decay of correlations for couplings of different qubits to the environment. For the random matrix model discussed above, this is trivially fulfilled. Yet, integrable environments are commonly used [18] and one may wonder whether Eq. (7) can hold in such a context. We shall therefore study a dynamical model where a few qubits are coupled to an environment represented by a kicked Ising spin chain using identical coupling operators for all qubits. In this model, the variation of the angle of the external kicking field allows the transition from a "quantum chaotic" to an integrable Hamiltonian for the environment [14]. The Hamiltonian of the chain is given by

$$H^{(e)} = \sum_{i=0}^{L-1} \sigma_x^{(e,i)}\sigma_x^{(e,i+1)} + \boldsymbol{\delta}(t)\sum_{i=0}^{L-1} b\cdot\sigma^{(e,i)}, \tag{11}$$

where $\boldsymbol{\delta}(t) = \sum_{n\in\mathbb{Z}}\delta(t-n)$ (i.e., time is measured in units of the kick period), L the number of spins in the environment, $\sigma^{(e,i)} = (\sigma_x^{(e,i)}, \sigma_y^{(e,i)}, \sigma_z^{(e,i)})$ the Pauli matrices of spin i, and b the dimensionless magnetic field kicking the chain. We close the ring requiring $\sigma^{(e,L)} \equiv \sigma^{(e,0)}$. The Hamiltonian of the QM is $H^{(\mathrm{QM})} = \boldsymbol{\delta}(t)\sum_i b\cdot\sigma^{(\mathrm{QM},i)}$, where $\sigma^{(\mathrm{QM},i)}$ is defined similarly as for the environment. The coupling is given by $\lambda V = \lambda\sum_i \sigma_x^{(\mathrm{QM},i)}\sigma_x^{(e,j_i)}$. The j_i's define the positions where the qubits of the QM are coupled to the spin chain. This model allows for efficient numerics because it is semiseparable and at each time step can be decomposed into unitary evolution of one or two qubits. Equivalently we could use kicked Ising couplings and a time-independent field [19].

To implement a chaotic environment, we use $b = (0.9, 0.9, 0)$. For these values the quantum-chaos properties have been tested extensively [20]. We use a ring consisting of $L = 12$ spins for the environment and 4 additional spins for the qubits of the QM. The coupling strength is $\lambda = 0.005$. In Fig. 1, we study purity decay when all four qubits are coupled to the same spin, to neighboring spins, and to maximally separated spins, $\vec{j} = (0, 3, 6, 9)$. The initial state is the product of a GHZ state in the QM and a random pure state in the environment. We compare the results with Eq. (7) (thick line) obtained from simulations of the spectator configuration (thin solid line). Coupling the spectator to different positions in the chain yields near identical results for $P(t)$, so we can see only one line. The figure demonstrates the validity of Eq. (7) for well-separated and hence independent couplings. For coupling to neighboring spins, decay is slightly faster while the sum rule does not hold if all qubits are coupled to the same spin.

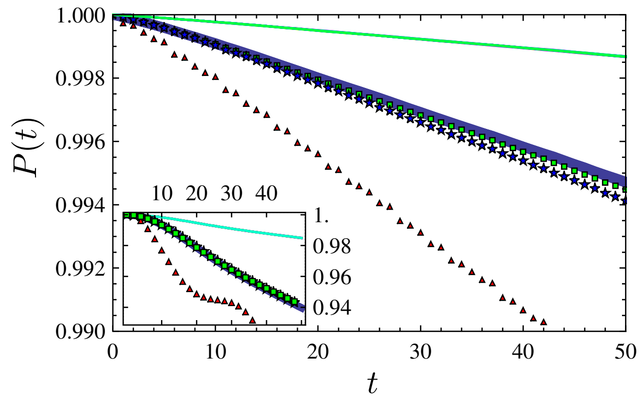

FIG. 1 (color online). Purity decay for a GHZ state is shown for the environment described by Eq. (11). The couplings are all qubits to one spin (red triangles), nearby spins (blue stars), and maximally separated spins (green squares). The theoretical result (thick line) is calculated in terms of the $P_{\mathrm{sp}}^{(i)}(t)$ (thin lines). The main figure corresponds to the case of a mixing environment and the inset to an integrable one.

A similar calculation for integrable environments with $b = (0, 1.53, 0)$ yields faster purity decay than in the mixing case as expected from general considerations in [21]. Nevertheless, the sum rule is again well fulfilled except if we couple all qubits to the same spin. This leads us to check the behavior of the correlation function

$$p[\tilde{V}_\tau^{(i)}\tilde{V}_{\tau'}^{(j)}\varrho_0\otimes\varrho_0] = \langle\psi_0|\tilde{V}_\tau^{(i)}\tilde{V}_{\tau'}^{(j)}|\psi_0\rangle. \tag{12}$$

This is the simplest and usually largest term in $A^{(i,j)}$, Eq. (6). Figure 2 shows this quantity for mixing and integrable environments in the first and second row, respectively. The first column shows the autocorrelation function ($i = j$). The second and third columns give the cross correlation function ($i \neq j$) when the qubits are coupled to the same or opposite spins, respectively. In the latter case correlations are always small thus showing that also in integrable situations our condition can be met. Nonvanishing correlations, as in Figs. 2(b) and 2(e), lead to deviations from the sum rule. The pronounced structure for the integrable environment, Fig. 2(e), may be associated to the oscillations of purity decay (inset of Fig. 1).

Another example where the conditions of our derivation are not met is the following. A Bose-Einstein condensate in which the n atoms have two immiscible internal states [22] could be interpreted as a QM. The symmetry of the wave function reduces the dimension of the Hilbert space and causes high correlations among the couplings to the envi-

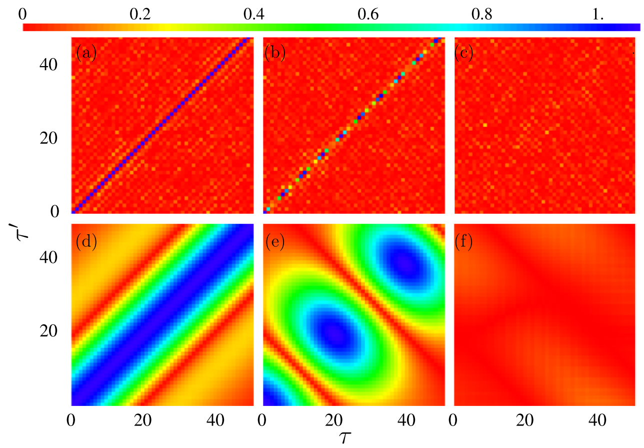

FIG. 2 (color online). The color-coded absolute value of the real part of $\langle\psi_0|\tilde{V}_\tau^{(i)}\tilde{V}_{\tau'}^{(j)}|\psi_0\rangle$ given in Eq. (12) as a function of τ and τ' for the chaotic [(a)–(c)] and the integrable [(d)–(f)] spin chain environment ($L = 8$). (a),(d) show the autocorrelation function ($i = j$), (b),(e) and (c),(f) show the cross correlation function for couplings to the same spin or opposite spins in the ring, respectively.

ronment. It is not surprising then that decoherence scales differently (as n^2).

We have calculated generic decoherence of an n-qubit quantum memory as represented by purity decay. For large purities it is globally given in linear response approximation by a sum rule in terms of the purities of the individual qubits entangled with the remaining register as spectator. This sum rule depends crucially on the absence or rapid decay of correlations between the couplings of different qubits in the interaction picture. We prove that this is fulfilled by sufficiently chaotic environments or couplings, but, using a spin chain model as an environment, we find that even if the latter is integrable correlations can be absent and the sum rule holds. While exceptions can be constructed we have a very general tool to reduce the decoherence of a QM of n qubits to the problem of a single qubit entangled with the rest of the QM. Furthermore, the result is equally valid if we perform local operations on the qubits. The sum rule also applies if the register is split into arbitrary sets of qubits. In fact, one and two qubit gates are known to be universal for quantum computation [1], so we lay the foundation for the computation of decoherence during the execution of a general algorithm. This extension only requires the knowledge of the decoherence of a pair suffering the gate operation while entangled with the rest of the register. As each gate is different, we will have to take a step-by-step approach, which at each step will only involve one and two qubit decoherence. The linear response approximation will be sufficient due to the high fidelity requirements of quantum computation.

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