Evolution of pairwise entanglement in a coupled *n*-body system

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We study the exact evolution of two noninteracting qubits, initially in a Bell state, in the presence of an environment, modeled by a kicked Ising spin chain. Dynamics of this model range from integrable to chaotic and we can handle numerics for a large number of qubits. We find that the entanglement (as measured by concurrence) of the two qubits has a close relation to the purity of the pair, and closely follows an analytic relation derived for Werner states. As a collateral result we find that an integrable environment causes quadratic decay of concurrence as well as of purity, while a chaotic environment causes linear decay. Both quantities display recurrences in an integrable environment.

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I. INTRODUCTION

Ever since the famous paper of Einstein, Podolski, and Rosen [1], entanglement and Bell states are at the center of interest of those concerned with the foundations of quantum mechanics. Recent experimental realizations of deterministic teleportation with photons [2,3] and ions [4,5], and a rising technical interest in quantum information [6] have stirred a great deal of interest in the decay of pairwise entanglement due to decoherence. Decoherence is one of the main problems in such applications [7–9], the other being systematic errors in the physical implementation of a logical or mathematical algorithm [10–12].

We choose two noninteracting qubits as a central system and start by considering their fully entangled state, i.e., a Bell pair. We then study the decay of entanglement within this pair due to decoherence. For our purposes we shall view decoherence as entanglement of our central system with some finite dimensional environment (e.g., see [13,14]) which will also be studied. We shall measure the former in terms of concurrence [15] and the latter in terms of purity (e.g., see [16]). While concurrence is trivially related to other measures of pairwise entanglement such as entanglement of formation [17], the choice of purity is rather arbitrary. It is based on the simplicity of the corresponding expression, which allows algebraic evaluation in some cases, while mathematically more satisfying measures such as, e.g., the von Neumann entropy are hard to handle due to the logarithm in the corresponding expression.

We shall focus on two points; first we wish to study how dynamics of the environment, i.e., its integrability or chaoticity influence the decay of the above quantities. Second, we shall analyze to what extent purity and concurrence may be related or, in other words, if there exists a universal relation between the two quantities at least in some approximation.

To implement such a program we need a model with flexible dynamics which allows efficient numerics for large Hilbert spaces. The fact that the central system should consist of two qubits makes a spin chain an attractive candidate. Indeed the kicked Ising spin chain, a model introduced by Prosen [18], was used to study the decay of fidelity and purity [19] in echo dynamics for integrable and chaotic (more precisely mixing [20]) as well as for intermediate cases. Results for purity in echo dynamics can be used for purity decay in standard forward dynamics if the perturbation is chosen as the coupling between the central system and the environment [19,21]. Apart from the flexibility concerning dynamics the main advantage of this model is the high computational efficiency that allows one to perform numerical calculations up to twenty qubits [22] and more on any good workstation. Yet the model in its original form does not allow for variable Ising interactions, and is thus not well suited outside of the field of echo dynamics. Experimental realizations of similar models have been proposed [23] and related studies appeared in [24].

In the present paper we shall first generalize this model to arbitrary interactions between the spins or qubits and we shall see that this does not affect the numerical efficiency of the model. We then use the generalized kicked Ising chain to study the evolution of purity and concurrence of the central system consisting of two spins in an initial Bell state multiplied by a random state evolving in environments with different dynamical properties. We want concurrence to be affected exclusively by the coupling to the environment. Therefore we chose noninteracting spins for the selected pair. For the environment it is sensible to consider random states to emulate a bath at fairly high temperature. Using a unitary time evolution of the total system and partial tracing over the environment we can then calculate concurrence and purity decay of the selected pair and discuss their behavior. Their dependence on the dynamics of the environment is significant. Yet the relation between purity and concurrence decay that is known for Werner states will be seen to hold approximately in all dynamics studied as we increase the size of the environment.

We shall first recall definitions of purity and concurrence and give the relation between both for Werner states. Next, we give details our model. In Sec. IV we shall display the

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behavior, both for the growth of entanglement as measured by the purity of the initial Bell pair and the decay of the concurrence of this pair. We shall then proceed to show the relation between concurrence and purity.

II. CONCURRENCE, PURITY, AND WERNER STATES

Concurrence (C) can be regarded as a good measure of entanglement for a two-qubit density matrix ρ [15,25]. It is defined as

$$C=\max\{0,\lambda_1-\lambda_2-\lambda_3-\lambda_4\}, \tag{1}$$

where λ_i are the eigenvalues of the matrix $\sqrt{\rho(\sigma_y\otimes\sigma_y)\rho^*(\sigma_y\otimes\sigma_y)}$ in nonincreasing order, (*) denotes complex conjugation in the computational basis, and σ_y is a Pauli matrix.

Purity is a measure of mixedness for a state characterized by a density matrix ρ in an arbitrary Hilbert space. It has a value of one for pure states and less than one in any other case. It is defined as

$$P(\rho)=\mathrm{Tr}\,\rho^2. \tag{2}$$

We use this measure instead of the usual von Neumann entropy (defined as $-\mathrm{Tr}\,\rho\log\rho$) since it is easier to handle from an algebraic point of view and both measures, albeit different, contain similar information.

Having a given value of purity in general does not fix the value of concurrence and vice versa. We can visualize the set of physical states in a plane with C and P in the axes. In Fig. 4 the gray region plus the line $\{(P,0)|P\in[1/4,1/3]\}$ represent the range of compatible values for concurrence and purity [26].

Let us define a general Bell state as

$$|\psi_{\text{Bell}}\rangle=\frac{|\mu_1\rangle|\mu_2\rangle+|\eta_1\rangle|\eta_2\rangle}{\sqrt{2}}, \tag{3}$$

where $\{|\mu_i\rangle,|\eta_i\rangle\}$ define an orthonormal basis for particle i. We shall now proceed to calculate the relation between purity and concurrence for a Werner state in which the entangled component is a general Bell state:

$$\rho_{\text{Werner}}=\frac{\alpha}{4}\mathbb{I}+(1-\alpha)|\psi_{\text{Bell}}\rangle\langle\psi_{\text{Bell}}|, \tag{4}$$

where α lies between zero and one.

Taking into account that purity and concurrence do not change under any independent particle unitary transformation, we may *choose* the computational basis so as to write $|\psi_{\text{Bell}}\rangle=(|00\rangle+|11\rangle)/\sqrt{2}$ in Eq. (4) yielding the explicit form of the density matrix

$$\rho_{\text{Werner}}=\begin{pmatrix}\frac{1}{2}-\frac{\alpha}{4} & 0 & 0 & \frac{1}{2}-\frac{\alpha}{2}\\ 0 & \frac{\alpha}{4} & 0 & 0\\ 0 & 0 & \frac{\alpha}{4} & 0\\ \frac{1}{2}-\frac{\alpha}{2} & 0 & 0 & \frac{1}{2}-\frac{\alpha}{4}\end{pmatrix}. \tag{5}$$

Then we calculate the concurrence [Eq. (1)] and purity [Eq. (2)], obtaining the exact expressions $P=1-3\alpha/2+3\alpha^2/4$ and $C=\max\{0,1-3\alpha/2\}$. Thus concurrence is given in terms of the purity by

$$C=\begin{cases}\dfrac{\sqrt{12P-3}-1}{2} & \text{if } 1/3<P\leqslant 1,\\ 0 & \text{if } 1/4\leqslant P\leqslant 1/3.\end{cases} \tag{6}$$

III. A GENERALIZED KICKED ISING MODEL

The Hamiltonian of the generalized kicked Ising model is

$$H=\sum_{j,k=0}^{L-1}J_{j,k}\sigma_j^z\sigma_k^z+\delta_1(t)\sum_{j=0}^{L-1}(b_j^{\perp}\sigma_j^x+b_j^{\parallel}\sigma_j^z),\quad \vec{\sigma}_L\equiv\vec{\sigma}_0, \tag{7}$$

where $\delta_1(t)$ represents an infinite train of Dirac delta functions with period one. The model thus consists of a set of spin-1/2 particles coupled to all other spins by an Ising interaction (first term) and periodically kicked by a site dependent tilted magnetic field (second term).

Our model differs from the one given in [18] by the fact that the coupling $J_{j,k}$ is between any pair of particles and has arbitrary strength, instead of $J_{j,k}=J\delta_{j+1,k}$, which couples nearest-neighbors in a chain with fixed strength. Throughout this paper we will again couple only first neighbors in a chain and use site independent magnetic field strength $b^{\perp}$, $b^{\parallel}$. Yet we shall keep the freedom of choosing arbitrary nearest-neighbor coupling $J_{j,k}=J_j\delta_{j+1,k}$.

To consider the situation described in the beginning of the paper we must weakly couple two spins, say spins "0" and "1," to the rest of the chain, which we will consider as the environment, i.e., J_1 and J_{L-1} are much smaller than the typical Ising interaction within the environment. We set $J_0=0$ in order to prevent any interaction between the spins in the central system. The fact that we keep the kick in the central system can represent local operations made by the "owners" of each of the qubits, and will not affect the values of concurrence and purity.

Rewriting the Hamiltonian in terms of the central system, environment, and interaction as $H=H_{\mathrm{c}}+H_{\mathrm{e}}+H_{\mathrm{ce}}$ the parts are given by

$$H_{\mathrm{c}}=\delta_1(t)\sum_{j=0}^{1}(b^{\perp}\sigma_j^x+b^{\parallel}\sigma_j^z), \tag{8}$$

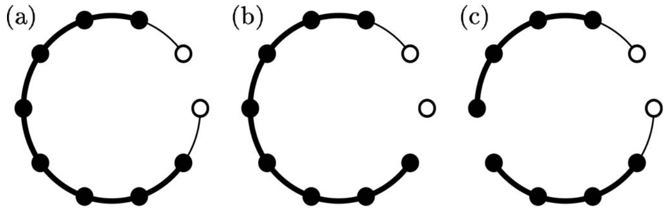

FIG. 1. Different configurations of the coupling of the system to the environment. The open circles represent the central system and the filled circles the environment. Thick/thin lines represent strong/weak interaction. In (a) we see both particles coupled to one environment, in (b) only one particle coupled to the environment and in (c) each particle coupled to two independent environments.

$$H_{\mathrm{e}} = \sum_{j=2}^{L-2} J_j \sigma_j^z \sigma_{j+1}^z + \delta_1(t) \sum_{j=2}^{L-1} (b^{\perp}\sigma_j^x + b^{\parallel}\sigma_j^z), \tag{9}$$

$$H_{\mathrm{ce}} = J_{L-1}\sigma_{L-1}^z \sigma_0^z + J_1 \sigma_1^z \sigma_2^z. \tag{10}$$

We shall consider three particular situations. The first one represents both particles coupled with equal strength to one environment, i.e., $J_0=0$ and $J_1=J_{L-1}=J_{\mathrm{c}}$, see Fig. 1(a). In the second one only one particle is coupled to the environment, i.e., $J_0=J_{L-1}=0$ and $J_1=J_{\mathrm{c}}$, see Fig. 1(b). Finally, we can couple each particle to independent environments, setting, e.g., $J_1=J_{L-1}=J_{\mathrm{c}}$ and $J_0=J_k=0$ for some $2<k<L-2$, see Fig. 1(c). The results for the three configurations are quite similar, and we shall discuss later the reasons to consider the three of them.

For the chain or chains considered as an environment we choose the following parameter values to emulate relevant dynamics. Setting the coupling $J_j=1$, the perpendicular component of the field $b_j^{\perp}=1.4$, and choosing the values 0.0, 0.8, and 1.4 for the parallel component of the field, we obtain integrable, mixed, and chaotic dynamical situations. The integrable and chaotic cases are discussed in Ref. [20] and Ref. [18], respectively. The intermediate case will be discussed later in terms of spectral statistics. It is pertinent to mention that these dynamical properties were obtained for cyclic chains, while the chains representing our environments are open. Yet for a large number of spins this is irrelevant.

The nonunitary evolution of the central system alone is calculated performing a unitary evolution of the whole system [yielding state $|\psi(t)\rangle$] and then performing the partial trace over the environment, that is

$$\rho(t) = \mathrm{Tr}_{\mathrm{env}}|\psi(t)\rangle\langle\psi(t)|. \tag{11}$$

We shall consider an initial condition

$$|\psi(t=0)\rangle = |\psi_{\mathrm{Bell}}\rangle \otimes |\psi_{\mathrm{Random}}\rangle, \tag{12}$$

i.e., a general Bell state [Eq. (3)] not entangled with the environment. Note that the reduced density matrix of the central system at this time is $|\psi_{\mathrm{Bell}}\rangle\langle\psi_{\mathrm{Bell}}|$ (a pure Bell state) which implies $C(\rho(0))=P(\rho(0))=1$.

Due to the two-body nature of the Hamiltonian we are able to calculate the time evolution of arbitrary initial conditions for up to 24 qubits on a good workstation, and calculations up to 17 qubits are carried out with ease on a laptop or PC. Therefore the model gives us a powerful tool to study a variety of phenomena, such as purity and concurrence decay. The essential difference to the calculations presented in [18,19] results from the fact that we are not restricted to echo dynamics, while we can use the same techniques, because we can now vary the coupling between the system and the environment.

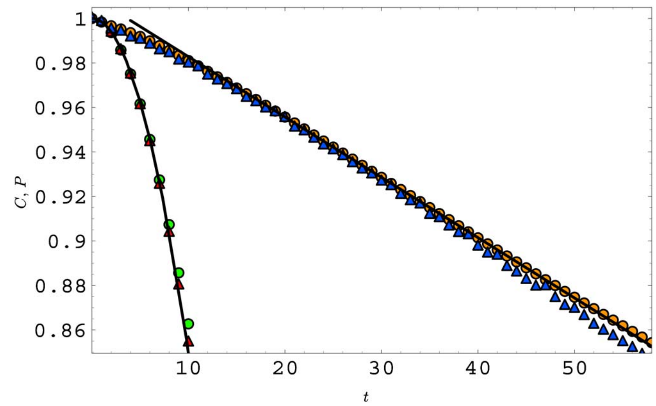

FIG. 2. (Color online) This figure shows the average evolution of concurrence (triangles) and purity (circles) for two different dynamics, with ten initial conditions given by Eq. (12). The environment is either chaotic ($b^{\parallel}=b^{\perp}=1.4$) or regular ($b^{\parallel}=0$ and $b^{\perp}=1.53$). In the chaotic case, both purity and concurrence show a linear dependence on time (see the black line which corresponds to a fit) whereas in the integrable one, both quantities exhibit a quadratic decay (see the fit to a curve of the kind $1-\gamma t^2$). We have 17 qubits and set $J_{\mathrm{c}}=0.02$.

Note that the ease to handle numerics is given for the fully generalized Hamiltonian Eq. (7). Allowing interaction between all qubits will only increase the effort by a factor corresponding to the number of qubits squared, i.e., the effort increases polynomially with the number of qubits. Choosing nearest-neighbor interactions in a lattice of two dimensions is another interesting option and so is the possibility of using a larger central system or more complicated interactions between the central system and the environment only.

IV. THE EVOLUTION OF CONCURRENCE AND PURITY

We now present the results of our numerical calculations of both concurrence and purity of the selected pair of spins as a function of time. We first concentrate on configuration (a) and on the short time behavior.

Figure 2 shows the time evolution of concurrence (triangles) and purity (circles) of the selected pair of spins for the initial state consisting of a Bell pair coupled to random states for the environment. We choose the environment to be mixing in one case and integrable in the other using the corresponding parameters from Ref. [18] as mentioned above. The mixing environment leads to near linear decay for some time interval, while the integrable environment leads to a quadratic decay. We show only high levels of

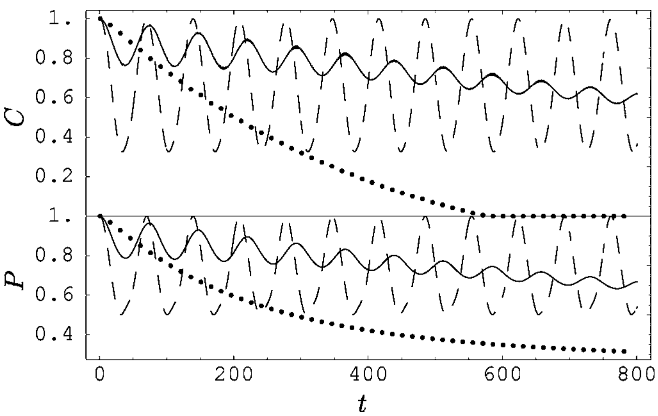

FIG. 3. Long time behavior of C (top) and P (bottom) for an arbitrary initial condition in integrable ($b^{\|}=0$, $b^{\perp}=1.53$), nonergodic and nonintegrable ($b^{\|}=0.8$, $b^{\perp}=1.4$), and fully chaotic ($b^{\|}=1.4$, $b^{\perp}=1.4$) regimes in dashed, continuous, and dotted lines, respectively. We used 17 qubits and $J_c=0.02$.

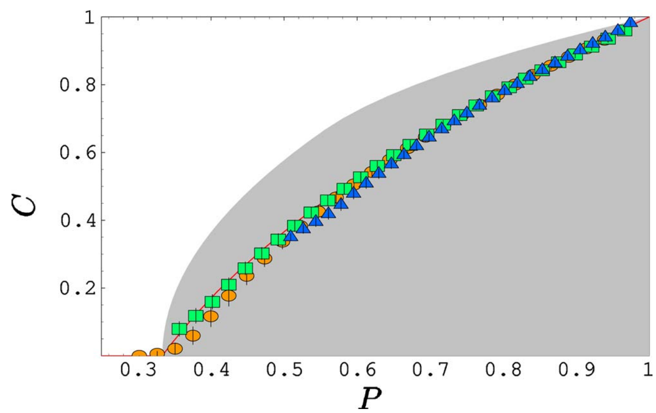

FIG. 4. (Color online) This figure shows the evolution of the system in the (C,P) plane, for different dynamical regimes; integrable, intermediate, and chaotic, in blue triangles, green squares, and orange circles, respectively. We use the same parameters as in Fig. 3. We use ten initial conditions and a total time of 3000 steps. The red line shows the relation for the Werner states in Eq. (6).

purity and concurrence; the continuation can be seen in the next figure (Fig. 3), including revivals for the integrable case.

This is not surprising in the case of purity, as the linear response result derived in [19] for the decay of purity in echo dynamics trivially translates to purity in forward time evolution, if we consider the coupling, Eq. (10), to be the perturbation of the echo; the fact that this behavior does not set in at very short times for the chaotic case results from a finite time of decay of the system-specific correlations of the perturbation, i.e., in our case the coupling in the uncoupled basis. This is also in keeping with the linear response calculations and numerics for entanglement production given in Ref. [27]. The result is not in contradiction to contrary findings for coherent states [28,29] for integrable systems as these have decay times governed by a different $\hbar$ dependence [29], though the quadratic dependence on time still holds.

With the first configuration, i.e., an environment of 15 spins coupled at each end to one of the selected spins that form the central system, ten states are chosen at random as in Eq. (12), and their averaged behavior is plotted in the figure.

We now proceed to look at the long time evolution of concurrence and purity again for configuration (a). In Fig. 3 we show both concurrence and purity for integrable, intermediate, and chaotic situations. Ensemble averaging is no longer necessary at this scale as fluctuations are small, and the curves represent the evolution of a single initial condition chosen randomly. The recurrences in the integrable case are almost complete for both quantities and their synchronization is manifest, although the autocorrelation function of the full system (not shown) drops fast and reaches negligible values even after the first few kicks. In the intermediate case, partial recurrences are visible on top of an average decay which becomes dominant in the chaotic case. The intermediate case was chosen to yield an intermediate behavior for the spectral statistics of the environment; the chosen parameters correspond to a Brody parameter [30] of approximately 0.33. This parameter fits a nearest-neighbour spacing distribution in the Floquet spectrum that shows a marked level repulsion, but is still quite far from the distribution we expect and get for the chaotic case. The Brody parameter does not characterize the behavior of the intermediate system uniquely, but the nongeneric features typical of intermediate structures mainly affect the period of oscillation and not the smooth decay. It may be worthwhile mentioning that in the integrable case we use the fact that the model is solvable rather than spectral statistics as criterion, because random spectra are rarely reached under any circumstances. It is also worthwhile to note that the oscillations of purity imply increase in some cross-correlation function [31].

For the different configurations proposed in Fig. 1 the behavior is qualitatively similar, hence we do not show figures. If the coupling strength is rescaled properly, differences are visible mainly for small environments, and otherwise would barely show in the equivalent figures.

V. A RELATION BETWEEN CONCURRENCE AND PURITY

The rather similar pictures emerging for concurrence and purity suggest that a simple relation between the two might emerge. We plot in Fig. 4 concurrence versus purity for 17 qubits, averaging purity for a given concurrence, again over ten initial conditions of the form given in Eq. (12). We plot the results for the integrable, the chaotic, and an intermediate choice of the environment. Remarkably we find that the three cases coincide within statistical error and that they agree with the relation given in Eq. (6) for Werner states, though they definitely are *not* Werner states; the latter fact was checked directly by considering the eigenvalues of the density matrix. We thus find that for a random environment on average the relation (6) holds though it was derived without dynamics.

Due to self-averaging this seems also to occur typically for an individual random state of the type proposed in Eq. (12) as the number of qubits increases; furthermore, the range over which the relation holds also increases with the size of the environment, as we can see in Fig. 5, where the

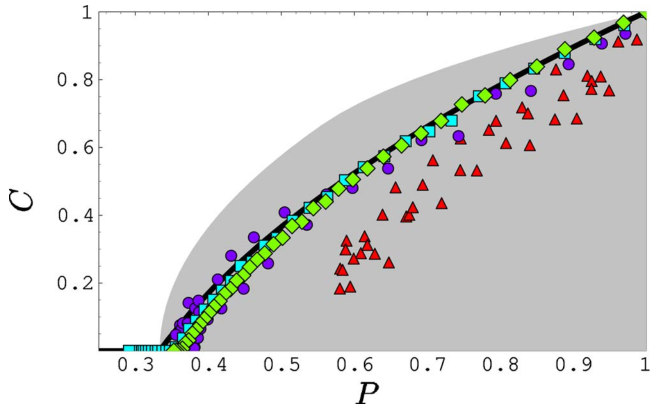

FIG. 5. (Color online) This figure shows the evolution for one initial condition in time steps of 15 up to 600 time steps in the (C,P) plane, in the chaotic regime. The number of qubits in the reservoir is varied, the red triangles, purple circles, blue squares, and green rhombus correspond to 4, 8, 12, and 16 qubits. The picture suggests that increasing the size of the environment improves the Werner state approximation. Similar results are obtained in the regular regime.

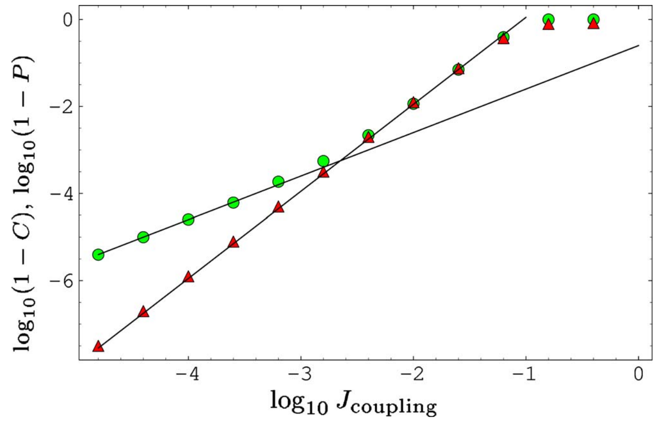

FIG. 6. (Color online) This figure shows the average behavior of C (in green circles) and P (in red triangles) with the perturbation parameter, J_c, for 20 time steps. We are in the chaotic regime with 22 qubits. The results are averaged over 20 initial conditions. Straight lines have slope 1 and 2. Note that purity behaves quadratically for small couplings, whereas concurrence also has a small linear component, which is a finite size effect. We observe the same scaling in other dynamical regimes.

relation is plotted for different numbers of qubits.

Again all results presented are calculated for the first of the three configurations shown in Fig. 1. The results for the other configurations are quite similar. We therefore do not show them, nevertheless they are of interest. For the second configuration one partner of the initial Bell pair is, dynamically speaking, a spectator, yet the evolution of concurrence and purity of the pair are marginally affected. The last case is more of an instructive toy model. Here we have two uncoupled environments, and we can start with a random state in each of these. The purity of the uncoupled subsystems will remain unchanged, but the purity and concurrence of the initial Bell pair will decay. Thus one might consider seeing a paradox, but this is not the case; the entanglement of the pair simply is spread over all of the system with time. Though the three configurations are physically quite different and the individual behavior of purity and concurrence is slightly affected, the relation between the two is entirely robust.

It remains to discuss how the coupling strength influences our results. As expected, stronger coupling accelerates the decay of both purity and concurrence. In Fig. 6 we show the dependence of concurrence and purity after 20 time steps on the coupling strength for a chaotic case. The double logarithmic scale immediately reveals two power laws for concurrence; J_c^{-2} for strong coupling and J_c^{-1} for weak coupling. Purity, on the other hand, depends on the coupling as J_c^{-2}, except for the largest couplings where saturation is reached. The crossover for concurrence, where the two curves split, is observed to move to ever weaker couplings as the size of the environment increases. The strength of the linear dependence scales approximately as $N^{-1/4}$, where N is the dimension of the Hilbert space. We thus conclude that the transition from quadratic to linear behavior is a finite size effect that nevertheless can be important. This implies that the relation of Werner states will not hold for small environments or very small perturbations.

VI. CONCLUSION AND OUTLOOK

We have analyzed the loss of concurrence and of purity of a pair of qubits in a Bell state coupled to chaotic, integrable, and mixed environments. We found these quantities after a transient to decay linearly for the chaotic case. For the integrable case we observed quadratic decay with strong revivals. The initial decay is much faster than for the chaotic case. The mixed case displays oscillations superimposed to a linear decay. We find that purity and concurrence in all cases behave in a qualitatively similar way. More surprisingly, the relation between the two for sufficiently large environments follows an analytic expression we get for Werner states quite closely, although the specific dynamics does not produce Werner states.

To perform these calculations we have generalized the model of a kicked Ising chain, such that couplings can be varied arbitrarily and indeed the chain character can be abandoned. We have used this model to calculate the decay of purity and of concurrence for variable coupling strength and size of the environment. The great flexibility and numerical efficiency of this model allows us to consider other applications. These include other initial states for the present system, multipartite entanglement, as well as similar situations for echo dynamics.

It will obviously be important to check how general our findings are concerning the relation between purity and concurrence. For that purpose we intend to analyze a random matrix model for chaotic dynamics. In the integrable case other examples should be considered, but as usual for integrable systems, we expect that some will follow the typical behavior and others not. These may provide an insight into the conditions of the surprisingly solid relation to the behavior of Werner states, but they may also provide special conditions for increased stability.

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[1] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).

[2] D. Bouwmeester, J. Pan, K. Mattle, M. Eibl, H. Weinfurter, and A. Zeilinger, Nature (London) **390**, 575 (1997).

[3] A. Furusawa, J. L. Sørensen, S. L. Braunstein, C. A. Fuchs, H. J. Kimble, and E. S. Polzik, Science **282**, 706 (1998).

[4] M. Riebe, H. Häffner, C. F. Roos, W. Hänsel, J. Benhelm, G. P. T. Lancaster, T. W. Körber, C. Becher, F. Schmidt-Kaler, D. F. V. James, *et al.*, Nature (London) **429**, 734 (2004).

[5] M. D. Barrett, J. Chiaverini, T. Schaetz, J. Britton, W. M. Itano, J. D. Jost, E. Knill, C. Langer, D. Leibfried, R. Ozeri, *et al.*, Nature (London) **429**, 737 (2004).

[6] A. Poppe, A. Fedrizzi, T. Loruenser, O. Maurhardt, R. Ursin, H. Boehm, M. Peev, M. Suda, C. Kurtsiefer, H. Weinfurter, *et al.*, Opt. Express **12**, 3865 (2004).

[7] M. B. Plenio and P. L. Knight, Proc. R. Soc. London, Ser. A **453**, 2017 (1997)

[8] B. J. Dalton, J. Mod. Opt. **50**, 951 (2003).

[9] R. J. Hughes, D. F. V. James, E. H. Knill, R. Laflamme, and A. G. Petschek, Phys. Rev. Lett. **77**, 3240 (1996).

[10] G. L. Celardo, C. Pineda, and M. Žnidarič, Int. J. Quantum Inf. **3**(3), 441 (2005).

[11] L.-A. Wu and D. A. Lidar, Phys. Rev. Lett. **91**, 097904 (2003).

[12] G. P. Berman, D. I. Kamenev, R. B. Kassman, C. Pineda, and V. I. Tsifrinovich, Int. J. Quantum Inf. **1**, 51 (2003).

[13] T. Gorin and T. H. Seligman, J. Opt. B: Quantum Semiclassical Opt. **4**, S386 (2002).

[14] K. Furuya, M. C. Nemes, and G. Q. Pellegrino, Phys. Rev. Lett. **80**, 5524 (1998).

[15] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).

[16] W. Zurek, Phys. Today **44**, (10) (1991).

[17] C. H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, Phys. Rev. A **54**, 3824 (1996).

[18] T. Prosen, Phys. Rev. E **65**, 036208 (2002).

[19] T. Prosen and T. H. Seligman, J. Phys. A **35**, 4707 (2002).

[20] T. Prosen, Atmos. Technol. **139**, 191 (2000).

[21] T. Gorin, T. Prosen, T. H. Seligman, and M. Žnidarič (to be published).

[22] C. Pineda, R. Schaefer, T. Prosen, and T. H. Seligman (to be published).

[23] J. P. Barjaktarevic, G. J. Milburn, and R. H. McKenzie, Phys. Rev. A **71**, 012335 (2005).

[24] A. Lakshminarayan and V. Subrahmanyam, Phys. Rev. A **71**, 062334 (2005).

[25] T.-C. Wei, K. Nemoto, P. M. Goldbart, P. G. Kwiat, W. J. Munro, and F. Verstraete, Phys. Rev. A **67**, 022110 (2003).

[26] W. J. Munro, D. F. V. James, A. G. White, and P. G. Kwiat, Phys. Rev. A **64**, 030302(R) (2001).

[27] H. Fujisaki, T. Miyadera, and A. Tanaka, Phys. Rev. E **67**, 066201 (2003).

[28] W. H. Zurek and J. P. Paz, Physica D **83**, 300 (1995).

[29] T. Prosen and M. Žnidarič, J. Phys. A **35**, 1455 (2002).

[30] T. A. Brody, Lett. Nuovo Cimento Soc. Ital. Fis. **7**, 482 (1973).

[31] T. Prosen, T. H. Seligman, and M. Žnidarič, Phys. Rev. A **67**, 062108 (2003).