Bell pair in a generic random matrix environment

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Two noninteracting qubits are coupled to an environment. Both coupling and environment are chosen as random matrices to obtain generic results. The initial state of the pair ranges from a Bell state to a product state. Decoherence of the pair is evaluated analytically in terms of purity; Monte Carlo calculations confirm these results and also yield concurrence of the pair. Entanglement within the pair accelerates decoherence. Numerics displays the relation between concurrence and purity known for Werner states. A closed albeit heuristic formula for concurrence decay ensues.

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I. INTRODUCTION

Decoherence is a subject of increasing interest in basic research [1–3] as well as in the context of stability of quantum information processes [4,5]. It is usually discussed in a context where the environment has a continuous spectrum or at least a very high spectral density. This implies that one relevant time scale—namely, Heisenberg time—moves off to infinity and all effects associated with this scale are invisible. Yet any experiment exploring the increase of entanglement of some system with a microscopically controlled environment will take place in a different setting. We shall present a *generic* framework that yields information on the scale of the Heisenberg time and reproduces the standard stochastic results at longer times. Random matrix theory [6] (RMT) models are excellent candidates to fulfill these conditions. Their surprising breadth of applications (see, e.g., the review [7] and the topical volume [8]) indicates generosity. Also RMT models that relate to the Caldeira-Legget model [9] or describe fast decoherence in strongly coupled systems [10] have been discussed.

The evolution of entanglement within a pair of qubits or spin-1/2 particles as well as of its coherence under the influence of an environment is paradigmatic for the stability of teleportation [11] and indeed for any quantum information process [4]. The environment(s) as well as the couplings will be modeled by matrices from one of the classical ensembles [12]. In the present article we shall concentrate on the Gaussian unitary ensemble (GUE), which describes time-reversal breaking dynamics, mainly because it provides the simplest analytics. The model we use is inspired by one developed for the evolution of fidelity [13] where it successfully describes experiments [14]. It was extended to give the evolution of echo-purity [15]; analytical results were obtained by a Born expansion in the interaction picture. Treating the coupling to the environment as the perturbation this tool can be used to study decoherence.

Purity, due to its simple analytic structure, is a particularly useful tool to describe the evolution of decoherence of a subsystem—i.e., its entanglement with the environment [1]. Concurrence provides a measure for the degree of entanglement within a pair of qubits [16]. We apply the RMT model to both measures. For purity we obtain analytic results along the lines mentioned above, and for concurrence we perform numerical studies. We find that purity of an entangled pair decays faster than purity of a product state, but we shall be able to go one step further. Numerically we show that the relation of purity to concurrence demonstrated for a specific dynamical model [17] holds for the random matrix model. Thus we can expect this behavior to be typical. This relation agrees numerically with the analytic one for Werner states. Combining the two results we give a closed, though heuristic, expression for concurrence decay. Both quantities and thus their relation are accessible by quantum tomography in experiments with trapped ions or atoms, where interaction with a controlled environment is feasible [18]. Thus the results are susceptible to experimental verification with state of the art techniques.

Concurrence of a density matrix ρ representing the state of a pair of qubits is defined as

$$C(\rho) = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}, \tag{1}$$

where λ_i are the eigenvalues of the matrix $\sqrt{\rho(\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y)}$ in nonincreasing order, $(*)$ denotes complex conjugation in the computational basis, and σ_y is a Pauli matrix. Purity is defined as

$$P(\rho) = \mathrm{Tr}\, \rho^2. \tag{2}$$

II. THE MODEL

We study dynamics on a Hilbert space with the structure $\mathcal{H} = \mathcal{H}_1^{\mathrm{q}} \otimes \mathcal{H}_1^{\mathrm{e}} \otimes \mathcal{H}_2^{\mathrm{q}} \otimes \mathcal{H}_2^{\mathrm{e}}$, where $\mathcal{H}_i^{\mathrm{q}}$ indicates (two-dimensional) qubit spaces, while $\mathcal{H}_i^{\mathrm{e}}$ will indicate N-dimensional environments. We consider unitary dynamics on the entire space and obtain nonunitary dynamics for the qubits by partial tracing over the environment(s). To consider the effect of the environment on the pair of qubits alone, we cannot allow any interaction within the pair but only interactions with the environment(s). For convenience we also

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neglect any possible evolution for each qubit individually, which is not induced by the coupling to the environment. The latter is nonessential to our argument, but simplifies the analytic treatment. We thus use the Hamiltonian

$$H = H_1^{\mathrm{e}} + H_2^{\mathrm{e}} + \lambda_1 V_1^{\mathrm{e,q}} + \lambda_2 V_2^{\mathrm{e,q}}. \tag{3}$$

The first two terms correspond to dynamics of the environments. The next two represent the coupling of the qubits to the corresponding environment. To obtain further simplification, we consider one of the qubits as a *spectator*, i.e., we assume that it has no coupling to an environment ($\lambda_2=0$). One environment becomes irrelevant, and we obtain the effective Hamiltonian

$$H_\lambda = H_1^{\mathrm{e}} + \lambda V_1^{\mathrm{e,q}}. \tag{4}$$

Note that we do consider entanglement with the spectator. This yields the *simplest* Hamiltonian for which we can analyze the effect of an environment on a Bell pair. The environment Hamiltonian H_1^{e} will be chosen from a classical ensemble [12] of $N\times N$ matrices and the coupling $V_1^{\mathrm{e,q}}$ from one of $2N\times 2N$ matrices. As usual, the GUE ensemble, which represents time-reversal invariance breaking dynamics, is easier to handle analytically than the Gaussian orthogonal one. Here we focus on the former. Evolution of both purity and concurrence of the pair of qubits can readily be simulated in a Monte Carlo calculation, and due to the simple structure of purity, it is possible to compute this quantity analytically in linear response (LR) approximation.

III. PURITY DECAY OF A BELL PAIR

The evolution operator is $U_\lambda=\exp[-\imath H_\lambda t]$, so the density matrix in Eqs. (1) and (2) is

$$\rho(t) = \mathrm{Tr}_{\mathrm{env}} U_\lambda |\psi(0)\rangle\langle\psi(0)| U_\lambda^\dagger, \tag{5}$$

where $|\psi(0)\rangle$ is the initial state of the system. Since U_0 is a local operation in the environment, it will not affect the value of ρ. Thus we can equally evolve with $U_0^\dagger U_\lambda$ instead of U_λ alone. It is convenient to use $U_0^\dagger U_\lambda$ since for small λ this operator remains in some sense near to unity for longer times.

To calculate the value of purity, we take the following averages and approximations. First we expand the echo operator $(U_0^\dagger U_\lambda)$ as a Born series requiring small λ and/or short times. We average both $V_1^{\mathrm{e,q}}$ (which will be called V from now on) and H_1^{e} over the appropriate GUE. Finally we average the initial state and obtain Eq. (13). This is the same scheme as the one followed in Ref. [13] for fidelity decay, though details are more complicated [10] due to the partial traces.

Following [13] we write the Born series to second order as

$$U_0^\dagger U_\lambda \approx \mathbb{1} - \imath\lambda I(t) - \lambda^2 J(t), \tag{6}$$

with

$$I(t) = \int_0^t d\tau \tilde{V}(\tau), \quad J(t) = \int_0^t d\tau \int_0^\tau d\tau' \tilde{V}(\tau)\tilde{V}(\tau'). \tag{7}$$

Here $\tilde{V}(t)$ is the coupling operator in the interaction picture: $\tilde{V}(t)=U_0^\dagger V U_0$. Writing $|\psi(0)\rangle=\sum_{\mu=1}^{4N} x_\mu|\mu\rangle$ and using Eq. (6), purity reads as

$$P(t) \approx 1 - \lambda^2(\mathrm{Re}\, A_J - A_1 - A_2 + \mathrm{Re}\, A_3), \tag{8}$$

where

$$A_J = 4 x_\mu x^*_{i'jk} x_{i'j'k'} x^*_{ij'k'} J_{ijk,\mu}(t), \tag{9a}$$

$$A_1 = 2 x_\mu x^*_\nu x_{i'j'k'} x^*_{ij'k'} I_{ijk,\mu}(t) I^*_{i'jk,\nu}(t), \tag{9b}$$

$$A_2 = 2 x_\mu x^*_{i'jk} x_{i'j'k'} x^*_\nu I_{ijk,\mu}(t) I^*_{ij'k',\nu}(t), \tag{9c}$$

$$A_3 = 2 x_\mu x^*_{i'jk} x_\nu x^*_{ij'k'} I_{ijk,\mu}(t) I_{i'j'k',\nu}(t) \tag{9d}$$

(summation over repeated indices is assumed). Here and in the sequel indices run as follows: Greek ones over the whole Hilbert space, i's over the environment, j's over the first qubit, and k's over the spectator qubit. Note that we use the natural notation for the indices of vectors in a space which is a tensor product of several spaces.

We now average the perturbation V over the GUE using $\langle V_{m,n}\rangle=0$ and $\langle V_{m,m'}V_{n,n'}\rangle=\delta_{m,n'}\delta_{m',n}$. Due to the unitary invariance of the GUE we may choose the basis that diagonalizes H_1^{e} yielding eigenvalues E_i. Then

$$\langle J(t)\rangle_{ijk,i'j'k'} = 2\delta_{ijk,i'j'k'} \int_0^t d\tau \int_0^\tau d\tau' \sum_{i''} e^{\imath(\tau'-\tau)(E_{i''}-E_i)}. \tag{10}$$

The matrix elements of the tensors $I\otimes I$ and $I\otimes I^*$, averaged, yield

$$\begin{aligned}\langle I_{ijk,lmn}(t_1) I_{i'j'k',l'm'n'}(t_2)\rangle &= -\langle I_{ijk,lmn}(t_1) I^*_{l'm'k',i'j'n'}(-t_2)\rangle \\ &= \delta_{ijk,l'm'n}\delta_{i'j'k',lmn'} \int_0^{t_1} d\tau \int_0^{t_2} d\tau' e^{\imath(E_i-E_l)(\tau-\tau')}.\end{aligned}$$

Next we average H_1^{e} over the GUE using that $\langle\sum_{i,i'} e^{\imath(E_i-E_{i'})t}\rangle=N[1+\delta(t/\tau_{\mathrm{H}})-b_2(t/\tau_{\mathrm{H}})]$ in the large-N limit. Here $b_2(t/\tau_{\mathrm{H}})$ is the form factor of the GUE [7] and τ_{H} the Heisenberg time, set to 2π throughout this article.

The initial state is a product of pure states for the qubit pair and the environment. For the latter we use a random initial state $\sum_i x_i|i\rangle$, constructed in the large-N limit, using complex random numbers x_i distributed according to a Gaussian centered around zero with width $1/\sqrt{N}$. For the pair of qubits we choose a completely general pure state. Since we still have the freedom to perform an arbitrary unitary local operation on each qubit due to the invariance properties of the GUE and the tensor product structure of the Hamiltonian, we select a basis in which the initial state for the two qubits has the form

$$|\varphi_\alpha\rangle = \cos\alpha|00\rangle + \sin\alpha|11\rangle, \quad \alpha \in [0,\pi/4]. \tag{11}$$

The degree of entanglement is characterized by α; in fact, $C(|\varphi_\alpha\rangle\langle\varphi_\alpha|)=\sin 2\alpha$. Hence our initial state can be written as $|\psi(0)\rangle=\Sigma_i x_i|i\rangle|\varphi_\alpha\rangle$. Neglecting higher-order terms in $1/N$, we obtain $\langle\langle A_J\rangle\rangle=2f(t)$ and $\langle\langle A_2\rangle\rangle=g_\alpha f(t)$ with

$$f(t)=\begin{cases} 2t\tau_{\rm H}+\dfrac{2t^3}{3\tau_{\rm H}} & \text{if } 0\leqslant t<\tau_{\rm H},\\ 2t^2+\dfrac{2\tau_{\rm H}^2}{3} & \text{if } t\geqslant\tau_{\rm H},\end{cases} \tag{12}$$

and $g_\alpha=\cos^4\alpha+\sin^4\alpha$. To leading order $\langle\langle A_1\rangle\rangle=\langle\langle A_3\rangle\rangle=0$. We obtain

$$P_{\rm LR}(t)=1-\lambda^2(2-g_\alpha)f(t). \tag{13}$$

From the explicit dependence of the result on g_α, we also see that the purity decay will be faster the more entangled the initial state is. The validity of this approximation is limited to large values of purity—i.e., short times or weak coupling. This is valuable for applications to quantum information, but we are interested in the dynamical picture as a whole and thus would like to obtain an expression valid for a wide range of physical situations. As a way to achieve this for fidelity decay, exponentiation of the leading term of the linear response (ELR) formula was proposed [19]. A similar approximation is taken here to obtain $P_{\rm ELR}(t)$. In order to calculate the appropriate formula, we must satisfy $P_{\rm ELR}(t)\approx P_{\rm LR}(t)$ for small t and consider correct asymptotics. These will be estimated as the purity after applying a totally depolarizing channel on one qubit to the two-qubit state, Eq. (11). The expected asymptotic value is $g_\alpha/2$, and the final expression is

$$P_{\rm ELR}(t)=\frac{g_\alpha}{2}+\left(1-\frac{g_\alpha}{2}\right)e^{[P_{\rm LR}(t)-1]/(1-g_\alpha/2)}. \tag{14}$$

This result is in excellent agreement with numerics as shown in Fig. 1. Here we obtain two different time regimes, an exponential one (Fermi golden rule) and a Gaussian one for weak perturbation. The time scale which defines the crossover between the two regimes is the Heisenberg time of the environment.

We have obtained these results using a spectator model on the one hand because it is the simplest one and on the other because it seems amusing and also quite characteristic of quantum mechanics that via entanglement, the mere presence of a noninteracting particle is relevant. Yet the general case may be more important and can be evaluated along the same lines for two independent environments. The form of the result is

$$P_{\rm LR}^{(d)}(t)=1-(2-g_\alpha)[\lambda_1^2 f_1(t)+\lambda_2^2 f_2(t)], \tag{15}$$

where $f_i(t)$ is identical to $f(t)$ as in Eq. (12), but using the Heisenberg time of $H_i^{\rm e}$ and the λ's are the couplings to each environment. Both particles interacting with a single environment implies taking $f_1(t)=f_2(t)=f(t)$ in the previous formula. The extension to longer times using exponentiation must take into account the fact that both qubits will decohere here; hence, the asymptotic value will be $1/4$:

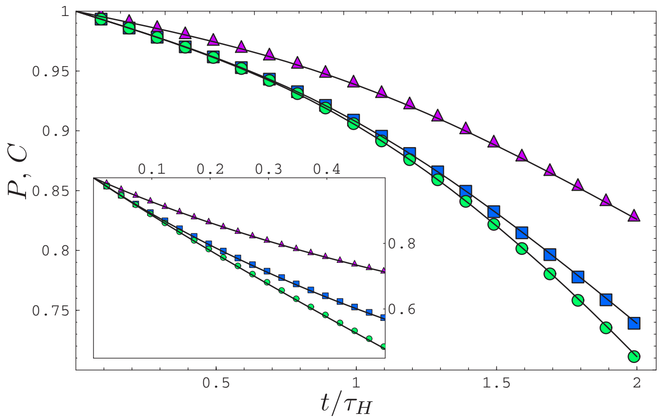

FIG. 1. (Color online) We show the evolution of purity for separable states (purple triangles), Bell states (green circles), and concurrence for Bell states (blue squares) in the crossover regime with significant decay before and after Heisenberg time. The lines show theoretical predictions given by Eqs. (14) and (17). The environment has dimension 2^{10}, and the perturbation strength is $\lambda=0.025$. In the inset we observe the Fermi golden rule regime for a larger perturbation $\lambda=0.1$.

$$P_{\rm ELR}^{(d)}(t)=\frac{1}{4}+\frac{3}{4}e^{(4/3)[P_{\rm LR}^{(d)}](t)-1}. \tag{16}$$

IV. ENTANGLEMENT DECAY OF A BELL PAIR

We have an approximate formula for the decay of purity of a Bell pair. What about concurrence? At this point we take up a result [17] for the behavior of a Bell pair coupled to a kicked spin chain [20]. For a wide range of situations the decay of a pure Bell state leads to purities and concurrences that closely follow those of a Werner state in a concurrence-purity (CP) diagram [17]. To test model independence and thus universality of this behavior we make the corresponding numerical simulations in the RMT model. We find that the Werner state CP relation is quite well fulfilled in the large-N limit, as can be seen in Fig. 2, where results for fixed coupling but different sizes of the RMT environment are shown. Studying other couplings allowed by the full Hamiltonian, Eq. (3), leads to similar results even if we are in different purity decay regimes. A partial explanation for this behavior can be found in [21], and a deeper study of this fact will be done in a future paper.

We have then the second relevant result of this article; namely, the relation of purity to concurrence for a noninteracting Bell pair interacting with a chaotic environment follows generically the curve of a Werner state. The importance of this statement is underlined by the fact that the actual state reached at any time is typically *not* a Werner state. This is tested by considering the spectrum of the density matrix, which should display a triple degeneracy for a Werner state. In fact a typical spectrum at $P=0.51$ is {0.692, 0.142, 0.110, 0.056}.

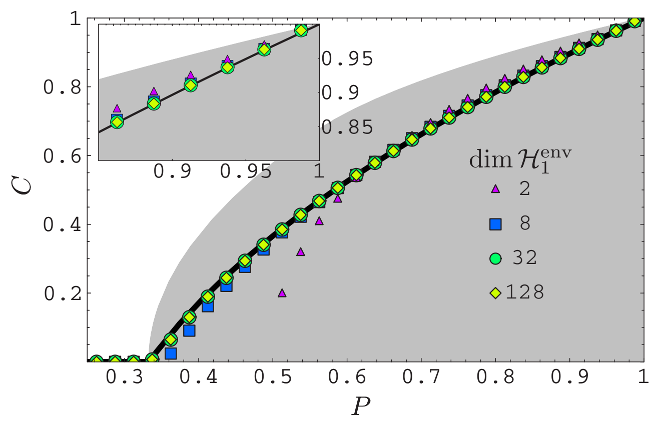

FIG. 2. (Color online) The relation between C and P for fixed coupling ($\lambda=0.3$) and variable environment size. We average over 10 realizations of the Hamiltonian (4) and (15) initial conditions. The gray area indicates the physical states and the line the concurrence-purity relation for Werner states. In the inset we observe short time deviations from this relation.

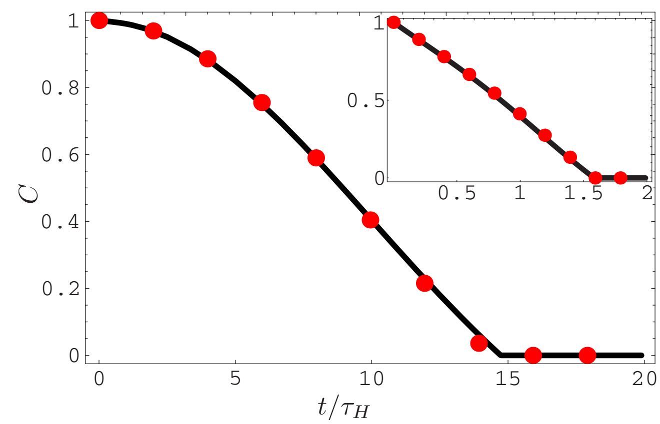

FIG. 3. (Color online) The average evolution of concurrence using an environment of dimension 128 and averaging over 10 Hamiltonians and 15 initial conditions. The black curve corresponds to Eq. (17). We show the Gaussian (δ=0.008) and the Fermi golden rule regimes in the inset (δ=0.07).

Having established the genericity of the known CP relation for a Werner state [17] we can insert the expression (14) for purity and obtain the heuristic result

$$C_{\text{ELR}}(t) = \max\left\{0, \frac{\sqrt{12P_{\text{ELR}}(t)-3}-1}{2}\right\} \tag{17}$$

for concurrence decay. In Fig. 3 we compare this relation to a Monte Carlo calculation and see that it describes the decay of concurrence of a Bell pair quite well. Notice that entanglement sudden death [22] is seen in Eq. (17) and Fig. 3. We inherit the encountered time regimes for purity. Since the exponential behavior can be obtained letting the Heisenberg time go to infinity, we retrieve results derived from a master equation approach [23].

V. CONCLUSIONS

Some points are worth mentioning: (a) We see significant deviations from the usual exponential decay at times of the order of the Heisenberg time as defined by the environment. Thus, if the spectrum of the environment becomes very dense and, correspondingly, the Heisenberg time moves off to infinity, we recover the usual stochastic result. (b) If the transition region can actually be seen, then the spectral stiffness of a chaotic environment has a small but significant stabilizing effect. The absence of spectral stiffness can be modeled by the so-called Poisson random ensemble [24] and leads to a faster decay. (c) We have limited our discussions to the GUE for two reasons. The simple expression of the form factor yields a concise final expression for purity decay. An additional advantage resulting from the unitary invariance of the coupling is that the final result is invariant under any local operation at each qubit. This is no longer guaranteed for orthogonal invariance only; the implications will be studied in a later paper.

Summarizing, we have developed a random matrix model for the evolution of a Bell pair interacting with a generic chaotic environment. Within this framework we derive the linear response approximation for the purity decay of a Bell pair and show that it differs significantly from decay of a product state of two qubits, even in the extreme case, where one of the qubits is only a spectator. Exponentiation extends the validity of this result far beyond its original reach. Monte Carlo calculations show that the relation between concurrence and purity, as obtained for Werner states, holds for RMT models and we thus have shown it to be generic. Based on these results we have obtained and tested a heuristic formula for the decay of concurrence of a noninteracting Bell pair.

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[1] W. Zurek, Phys. Today **44** (10), 36 (1991); D. Braun, F. Hacke, and W. T. Stunnz, Phys. Rev. Lett. **86**, 2913 (2001).

[2] J. von Neumann, *Mathematical Foundations of Quantum Mechanics* (Princeton University Press, Princeton, 1955).

[3] H. D. Zeh, e-print quant-ph/9506020.

[4] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2000).

[5] G. P. Berman, G. D. Doolen, R. Mainieri, and V. I. Tsifrinovich, *Introduction to Quantum Computers*, 1st ed. (World Scientific, Singapore, 1998).

[6] M. L. Mehta, *Random Matrices* 2nd ed. (Academic Press, San

Diego, 1991).

[7] T. Guhr, A. Mueller-Groeling, and H. A. Weidenmueller, Phys. Rep. **299**, 189 (1998).

[8] Special issue, J. Phys. A **36**(12, (2003).

[9] E. Lutz and H. A. Weidenmueller, Physica A **267**, 354 (1999).

[10] T. Gorin and T. H. Seligman, J. Opt. B: Quantum Semiclassical Opt. **4**, S386 (2002).

[11] M. Riebe *et al.*, Nature (London) **429**, 734 (2004); M. D. Barrett *et al.*, *ibid.* **429**, 737 (2004).

[12] È. Cartan, Abh. Math. Sem. Hamburg **11**, 116 (1935).

[13] T. Gorin, T. Prosen, and T. H. Seligman, New J. Phys. **6**, 20 (2004).

[14] R. Schäfer, H. J. Stockmann, T. Govin, and T. H. Seligman, *et al.*, Phys. Rev. Lett. **95**, 184102 (2005); , New J. Phys. **7**, 152 (2005); T. Gorin, T. H. Seligman, and R. L. Weaver, Phys. Rev. E **73**, 015202(R) (2006).

[15] T. Gorin, T. Prosen, T. H. Seligman, and M. ŽNidarič, Phys. Rep. **435**, 33 (2006).

[16] S. Hill and W. K. Wootters, Phys. Rev. Lett. **78**, 5022 (1997).

[17] C. Pineda and T. H. Seligman, Phys. Rev. A **73**, 012305 (2006).

[18] R. Blatt and H. Häffner (private communication).

[19] T. Prosen and T. H. Seligman, J. Phys. A **35**, 4707 (2002).

[20] T. Prosen, Phys. Rev. E **65**, 036208 (2002).

[21] M. Ziman and V. Buzek, Phys. Rev. A **72**, 052325 (2005).

[22] T. Yu and J. H. Eberly, Phys. Rev. Lett. **93**, 140404 (2004).

[23] F. Mintert *et al.*, Phys. Rep. **415**, 207 (2005).

[24] F.-M. Dittes, I. Rotter, and T. H. Seligman, Phys. Lett. A **158**, 14 (1991).